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ENHANCING PROGRAM EVALUATION AND REVIEW TECHNIQUE (PERT) FOR CONSTRUCTION PROJECT SCHEDULING WITH BAYESIAN UPDATING AND APPROPRIATE PROBABILITY DISTRIBUTIONS

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Abstract

The Program Evaluation and Review Technique (PERT) is a popular scheduling technique that takes advantage of the Beta distribution to present uncertainty in activity durations. This study presents an advanced PERT method with an improved Bayesian updating and improved assumed prior distributions, which better represent real-world projects. The method is backed with detailed mathematical proofs and derivations for a solid theoretical foundation. A numerical case study involving a 30-floor building construction project is used to compare the performance of traditional PERT, the Beta-improved Bayesian PERT, and the Log-Normal Bayesian PERT methods. In the example, the activities considered are Formwork, Rebar and Construction, Masonry, Mechanical-Electrical-Plumbing (MEP), and Finishing, which are the main activities in a construction project. The results show that the Beta-improved and the Log-Normal distributions are constructed successfully in the models with converging variance – an observation that delineates the uncertainty reduced along a real project's course. With enhanced functions, the PERT method can be utilized to support project decision-makers in scheduling and managing complex projects in reality.

Keywords: scheduling; PERT; Bayesian; log-normal; beta; simulation.

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1. Introduction

The Program Evaluation and Review Technique (PERT) is a popular scheduling technique that can take into account the uncertainty in project activities' duration estimation. Developed by the U.S. Navy in the late 1950s, PERT rather uses a probabilistic approach but not a deterministic approach to estimate task durations [1, 2]. Used together with the Critical Path Method (CPM) this method allows project managers to visualize the interdependencies of tasks and assess the overall project timeline, which is crucial in construction where delays can lead to significant cost overruns [3, 4]. However, PERT has obvious limitations. The first limitation is that it relies on the beta distribution to model activity durations – a technique that has been criticized for its oversimplified feature that fails to capture the real-world activities [5, 6]. Secondly, the original PERT often underestimates the average project duration but overestimates the variance and the result is that the outcomes are often not precise enough [6, 7]. This underestimation is even exacerbated in the context of real long and volatile projects [4, 8]. Thirdly, PERT assumes a three-point estimation for every activity. This assumption is too simplified and often fails to represent actual risks that usually occur in a right-skewed manner [9]. This study introduces an advanced PERT method, in which the Bayesian approach is used to

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update the activities as new information arrives, and the assumed distributions are enhanced with the Beta improved and the introduction of the Log-Normal distribution. The paper is structured as follows: a literature review of PERT's basics, limitations, and related studies to overcome its limitations; next, mathematical proofs and derivations to substantiate the theoretical advantages of the proposed methods; a numerical example with a simplified 30-floor building project is analyzed to validate the proposed methods; finally, a discussion of implication on construction scheduling and the conclusion finish the study.

2. Literature review on PERT

2.1. Limitations of traditional PERT

Although PERT has been used in many complex projects since its debut decades ago, it has obvious limitations. It is crucial to enhance the method first with a comprehensive review of PERT's limitations and previous efforts to address these limitations.

The first limitation is the assumption of the Beta distribution when PERT deals with activities' durations. The Beta distribution means that an activity duration has an optimistic (a), a most likely (m), and a pessimistic (b) time estimates to calculate expected durations and variances. These three estimates often fail to represent actual activity durations, especially when the data are skewed or have heavy and asymmetric tails [10]. As a result, this mismatch can lead to inaccurate total project duration and inadequate risk assessment [2].

Secondly, the estimates of activities' durations of PERT from the planning phase, stay static during the project's course, ignoring the arrival of new information. This lack of adaptability means that PERT cannot incorporate real-time performance data or respond to changes in project conditions, reducing its relevance and accuracy over time [11].

The next shortcoming is that the method uses simplified statistical elements such as expected durations and variances, which are not enough to represent the complexity of real-world projects [9]. A realistic statistical model must include many more factors so that it reflects real construction projects.

One of the most famous limitations of PERT is that it assumes that activity durations are independent, but not dependents on one another because of shared resources, environmental conditions, or other dependencies [12]. This limitation arguably prevents the successful management of the overall project risk, especially in complex and long ones [13].

Finally, even if practitioners want to update activity estimates when new information arrives, they do not have a tool backing them [1, 14]. When new information is unused, project managers cannot effectively adjust schedules and resourse in response to actual performance or emerging issues.

2.2. Attempts to address PERT's limitations

Of course the limitations of the traditional PERT have not been ignored completely: researchers and practitioners have proposed some changes to enhance the method's accuracy and effectiveness in project scheduling. One common direction has been to introduce other statistical distributions that better capture the characteristics of activity durations [15–17]. For example, the Triangular distribution has been suggested as a simpler alternative to the Beta distribution due to its ease of parameter estimation and its ability to model skewness. However, the Triangular distribution cannot represent the heavy-tailed behaviors often seen in project activities [18].

This heavy-tailed characteristics, when activities are influenced by multiplicative factors, is indeed solvable by the Log-Normal distribution because the function's graphs are right-skewed in their nature [19]. Some delays, such as waiting for construction permit or a lack in financial budget, can be as long as multiple times their expected values but they can be modeled by the Log-Normal distribution [7].

Another research direction has been to focus on taking advantage of Bayesian statistics to dynamically update activity durations to PERT as new information recognized [20, 21], so called Bayesian PERT. Bayesian statistics have been introduced in the management of construction projects in various fields, such as risk [22], quality [23], safety [24], decision-making [25], and scheduling management [26]. Bayesian PERT is initially based on prior knowledge (or assumptions) obtained from historical data or experts, then keeps on refining itself in light of new evidence [27]. This approach is aligned with the last limitation mentioned above: providing practitioners a mechanism to update beliefs. Specifically with the Beta distribution, researchers have proposed some Bayesian models to adjust the parameters of the distribution [20]. For example, hierarchical Bayesian models can treat uncertainty as multiple levels of variability, such as individual activities and overall project performance [27]. Another proposed distribution is Bayesian synthetic likelihood [28], a method for approximating likelihood functions in complex models, which can be used when traditional likelihood calculation are computationally infeasible. But this method relies heavily on the assumption that the synthetic likelihood accurately represents the true likelihood, which may not always be the case. Muller et al. explore Bayesian nonparametric methods, which allow for flexible modeling of data without assuming a fixed parametric form [29], but require more effort to implement and interpret than traditional parametric methods.

Regarding the inadequate risk assessment limitation in traditional PERT, some scholars have brought Monte Carlo simulation into the calculation to model the uncertainty and variability in activity durations [16]. When models are run many times (e.g., ten thousand or a million times), results form certain probabilistic distributions of the total project duration, from which practitioners can manage risk and plan for contingency better. This technique allows different scenarios, including extreme cases to be considered with quantified risks [30].

Another research direction is to combine PERT with other popular project management techniques (e.g., Critical Chain Method or Earned Value Management) to form hybrid models that can theoretically enhance scheduling accuracy and resource allocation [31–33]. As computational power increases exponentially, the development of more sophisticated algorithms (e.g., [34]) has been benefited as well. With advanced functions and user-friendly interfaces, these tools have been usable to most of project management practitioners in the industry.

Despite these advancements, literature also shows some research gaps. For example, the Beta distribution stills dominates the popularity over other more realistic but sophisticated distributions like Log-Normal, just because of its simplicity. The potential of Bayesian frameworks to dynamically adjust activity duration estimates from updated data is also underdeveloped. Furthermore, some studies (e.g., [1, 18]) have tried to address these gaps but mathematical proofs to theoretically prove the soundness and reliability in practical applications of the methods have not been presented in detailed. Finally, few studies have compared the performance of traditional PERT and that of proposed methods with examples to help readers visualize the efficiency of new methods.

2.3. Addressing the research gaps

To address PERT's incapability of updating its activity durations and its weakness in presenting right-skewed activities and risks in construction projects, the author proposes an advanced PERT method integrated with Bayesian statistics which can use the Log-Normal distribution or improved Beta parameters if practitioners decide to keep using this distribution. Mechanisms to dynamically update activity durations during the project course as new observations arrive are examined as well.

Throughout the development process, detailed mathematical proofs and derivations are provided to theoretically validate the robustness of proposed models. An example involving a 30-floor building under construction is used to: (i) depict the steps to use new methods, showing their practical application, and (ii) compare the results of traditional PERT with new methods to highlight their possible improvements over that of traditional PERT. The results of this study are expected to contribute to the body of knowledge in the project scheduling field, showing the flexibility, effectiveness, and applicability of these advanced Bayesian PERT methods.

3. Advanced Bayesian PERT method

3.1. Proposed approaches

To summarize the approach, the author incorporates Bayesian statistics in PERT for an updating mechanism while improving underlying distributions with the Log-Normal distribution and improved Beta parameters. The rationale for alternating distribution to Log-Normal is that it can model right-skewed events, which are most common in construction projects. The reason to enhance Beta parameters without urging to removal of the distribution in PERT is because of the simplicity of the estimates, and the Beta distribution can still model short and less complex projects efficiently. Upon successfully integrating appropriate distributions and the nature of Bayesian updating into PERT, new models can be used extensively by practitioners because they can obtain posterior distributions from continuously observed data hence making decisions with more confidence.

3.2. Beta distribution approach

a. Parameter estimation

Given the optimistic (a), most likely (m), and pessimistic (b) time estimates for an activity, the traditional PERT expected duration E and variance V are:

$$E = \frac{a + 4m + b}{6} \quad \text{and} \quad V = \left(\frac{b - a}{6}\right)^2 \tag{1}$$

To fit a Beta distribution $Beta(\alpha,\beta)$ to the activity duration, the duration T is standardized to $X \in [0, 1]$:

$$X = \frac{T - a}{b - a} \tag{2}$$

The mean μ and variance σ^2 of X are:

$$\mu = \frac{\alpha}{\alpha + \beta} \quad \text{and} \quad \sigma^2 = \frac{\alpha\beta}{(\alpha + \beta)^2\,(\alpha + \beta + 1)} \tag{3}$$

Using the method of moments, the author sets these parameters:

$$\mu = \frac{E - a}{b - a}; \quad \sigma^2 = \frac{V}{(b - a)^2} \tag{4}$$

Solving for α and β:

$$F = \frac{\mu\,(1 - \mu)}{\sigma^2} - 1; \quad \alpha = \mu F; \quad \beta = (1 - \mu)\,F \tag{5}$$

where F is a useful common factor.

b. Bayesian updating

After observing n activity durations $x_1, x_2, \ldots, x_n$, the posterior Beta parameters are:

$$\alpha' = \alpha + \sum_{i=1}^{n} x_i \tag{6}$$

$$\beta' = \beta + n - \sum_{i=1}^{n} x_i \tag{7}$$

The posterior mean and variance are also updated as:

$$\mu' = \frac{\alpha'}{\alpha' + \beta'} \tag{8}$$

$$\sigma'^2 = \frac{\alpha'\beta'}{(\alpha' + \beta')^2 (\alpha' + \beta' + 1)} \tag{9}$$

• Proof of variance reduction

- Theorem: In the Beta-improved Bayesian PERT method, the posterior variance decreases as the number of observations increases.

- Proof:

It is helpful to show that $\sigma'^2 < \sigma^2$ as n increases.

Consider the ratio of the posterior variance to the prior variance:

$$\frac{\sigma'^2}{\sigma^2} = \frac{\alpha'\beta'}{\alpha\beta} \times \left(\frac{(\alpha + \beta)^2 (\alpha + \beta + 1)}{(\alpha' + \beta')^2 (\alpha' + \beta' + 1)} \right) \tag{10}$$

As $n \to \infty$:

+ *Numerator Increase*: Both α' and β' increase due to added observations, but their product $\alpha'\beta'$ increases at a rate slower than the increase in their sum $\alpha' + \beta'$.

+ *Denominator Increase*: The term $(\alpha' + \beta')^2 (\alpha' + \beta' + 1)$ increases faster than $\alpha'\beta'$ due to the squared and cubic terms.

Therefore, the overall ratio $\frac{\sigma'^2}{\sigma^2}$ decreases as n increases, showing that the posterior variance σ'^2 is less than the prior variance σ^2. We can conclude that the uncertainty in the estimate decreases with more observations.

• Proof of convergence of posterior mean

- Theorem: The posterior mean $\mu' = \frac{\alpha'}{\alpha' + \beta'}$ converges to the sample mean $\overline{x}$ as $n \to \infty$.

- Proof:

The posterior mean estimated as:

$$\mu' = \frac{\alpha'}{\alpha' + \beta'} = \frac{\alpha + S_n}{\alpha + \beta + n} \tag{11}$$

Let us divide the numerator and denominator by n:

$$\mu' = \frac{\frac{\alpha}{n} + \overline{x}}{\frac{\alpha + \beta}{n} + 1} \tag{12}$$

As $n \to \infty$, it is obvious that $\mu' \to \overline{x}$, implying that with a large number of observations, the influence of the prior diminishes, and the estimate is dominated by the observed data.

3.3. Log-Normal distribution approach

a. Parameter estimation

For the Log-Normal distribution, $T \sim LogNormal\left(\mu, \sigma^2\right)$, the mean and variance are:

$$E[T] = e^{\left(\mu + \frac{\sigma^2}{2}\right)} \tag{13}$$

$$V[T] = \left(e^{\sigma^2} - 1\right) e^{\left(2\mu + \sigma^2\right)} \tag{14}$$

Similarly, $E[T]$ and $V[T]$ are expected and variance of the duration T correspondingly.
Using the PERT estimates E and V, the author solves for μ and σ^2:

$$\sigma^2 = \ln\left(1 + CV^2\right); \quad \mu = \ln(E) - \frac{\sigma^2}{2} \quad \text{whereby} \quad CV = \frac{\sqrt{V}}{E} \tag{15}$$

b. Bayesian updating

Assuming conjugate priors for μ and σ^2:
- Prior for μ: Normal distribution $N\left(\mu_0, \tau_0^2\right)$;
- Prior for σ^2: Inverse-Gamma distribution $\Gamma^{-1}(\alpha_0, \beta_0)$.

After observing n log-transformed durations $y_i = \ln t_i$:
- Posterior for σ^2:

$$\sigma'^2 \sim \Gamma^{-1}\left(\alpha_0 + \frac{n}{2}, \beta_0 + \frac{1}{2}\sum_{i=1}^{n}(y_i - \bar{y})^2 + \frac{n(\bar{y} - \mu_0)^2}{2\left(1 + n\tau_0^2\right)}\right) \tag{16}$$

- Posterior for μ:

$$\mu' = \frac{\dfrac{\mu_0}{\tau_0^2} + \dfrac{n\bar{y}}{\sigma'^2}}{\dfrac{1}{\tau_0^2} + \dfrac{n}{\sigma'^2}} \tag{17}$$

$$\tau'^2 = \frac{1}{\dfrac{1}{\tau_0^2} + \dfrac{n}{\sigma'^2}} \tag{18}$$

• Proof of variance reduction

- Theorem: In the Log-Normal Bayesian PERT method, the posterior variance decreases as the number of observations increases.
- Proof:

Consider Eq. (18): as n increases, the term $\dfrac{n}{\sigma^2}$ dominates $\dfrac{1}{\tau_0^2}$, so $\tau'^2 \approx \dfrac{\sigma^2}{n}$. This indicates that the posterior variance decreases inversely with n, implying increased precision in the estimate of μ as more data is collected.

• Proof of convergence of posterior mean

- Theorem: The posterior mean μ' converges to the sample mean $\bar{y}$ as $n \to \infty$.
- Proof:

The posterior mean is given by:

$$\mu' = \left(\frac{\mu_0}{\tau_0^2} + \frac{n\bar{y}}{\sigma^2}\right) \times \tau'^2 \tag{19}$$

Substituting τ'^2:

$$\mu' = \left(\frac{\mu_0}{\tau_0^2} + \frac{n\bar{y}}{\sigma^2}\right) \times \frac{1}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}} = \frac{\frac{\mu_0}{\tau_0^2} + \frac{n\bar{y}}{\sigma^2}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}} \tag{20}$$

As $n \to \infty$: $\frac{n}{\sigma^2}$ dominates $\frac{1}{\tau_0^2}$ and $\mu' \to \frac{\frac{n\bar{y}}{\sigma^2}}{\frac{n}{\sigma^2}} = \bar{y}$. Thus, the posterior mean converges to the sample mean $\bar{y}$ of the log-transformed data, demonstrating that with sufficient data, the prior's influence diminishes.

• Proof of reduction in posterior variance of activity duration

Consider $V[T]$ in Eq. (14): As the posterior estimates of σ^2 decrease with more observations (since σ^2 has an Inverse-Gamma posterior that becomes more concentrated), the term $e^{\sigma^2} - 1$ decreases. Additionally, as μ becomes more precise (variance τ'^2 decreases), the overall variance $V[T]$ decreases.

4. Numerical case study

4.1. Numerical example description

Let us consider the construction of a 30-floor building with the following sequential activities on each floor: Activity A: Formwork; Activity B: Rebar and Concrete; Activity C: Masonry; Activity D: Mechanical, Electrical, and Plumbing (MEP); Activity E: Finishing.

4.2. Activity time estimates

Prior estimates of typical activities are given in Table 1, for instance, by experts or history data.

Table 1. Time estimates (days) of typical activities

Activity Time Estimates	A	B	C	D	E
Optimistic (a)	3	4	5	6	4
Most likely (m)	5	6	7	8	6
Pessimistic (b)	8	10	12	14	9

4.3. Traditional PERT calculations

For each activity, the expected duration E and variance V are calculated (Table 2):

Total Expected Duration and Variance per Floor:

$$E_{Floor} = E_A + E_B + E_C + E_D + E_E = 34.5 \text{ days} \tag{21}$$

$$V_{Floor} = V_A + V_B + V_C + V_D + V_E = 5.52 \text{ days} \tag{22}$$

Total Project Duration for 30 Floors:

$$E_{Total,PERT} = 30 \times E_{Floor} = 1035 \text{ days} \tag{23}$$

$$V_{Total,PERT} = 30 \times V_{Floor} = 165.6 \text{ days}^2 \tag{24}$$

Table 2. Duration and Variance of activities (traditional PERT)

Expected Duration E and Variance V	A	B	C	D	E
$E = \dfrac{a+4m+b}{6}$ (day)	5.17	6.67	7.83	8.67	6.17
$V = \left(\dfrac{b-a}{6}\right)^2$ (day^2)	0.69	1	1.36	1.78	0.69

4.4. Beta-Improved Bayesian PERT method

a. Prior parameters

Using the method described above, the author calculates α and β for each activity.

Example for Activity A:

- Standardized mean:

$$\mu_A = \frac{E_A - a_A}{b_A - a_A} = \frac{5.17 - 3}{5} = 0.433 \text{ (days)} \tag{25}$$

- Standardized variance:

$$\sigma_A^2 = \frac{V_A}{(b_A - a_A)^2} = \frac{0.69}{25} = 0.0276 \text{ (days}^2\text{)} \tag{26}$$

- Common factor F_A:

$$F_A = \frac{\mu_A (1 - \mu_A)}{\sigma_A^2} - 1 = \frac{0.433 \times 0.567}{0.0276} - 1 \approx 8.9 \tag{27}$$

- Beta parameters:

$$\alpha_A = \mu_A F_A \approx 3.86, \beta_A = (1 - \mu_A) F_A \approx 5.04 \tag{28}$$

b. Bayesian updating

For each observed duration t_i of Activity A:

- Standardize:

$$x_i = \frac{t_i - a_A}{b_A - a_A} \tag{29}$$

- Update parameters:

$$\alpha' = \alpha + \sum x_i \tag{30}$$

$$\beta' = \beta + n - \sum x_i \tag{31}$$

- Compute posterior mean and variance.

4.5. Log-Normal Bayesian PERT method

a. Prior parameters

Using the method described above, the author estimates μ and σ^2 for each activity.

Example for Activity A:

- Coefficient of variation:

$$CV_A = \frac{\sqrt{V_A}}{E_A} = \frac{\sqrt{0.69}}{5.17} \approx 0.16 \tag{32}$$

- Estimate σ^2:

$$\sigma_A^2 = \ln\left(1 + CV_A^2\right) \approx \ln(1 + 0.0256) = 0.0253 \tag{33}$$

- Estimate μ:

$$\mu_A = \ln(5.17) - \frac{0.0253}{2} \approx 1.641 \tag{34}$$

b. Bayesian updating

For each observed duration t_i:
- Compute:

$$y_i = \ln t_i \tag{35}$$

- Update μ and σ^2 using the formulas in Section 3.3.

4.6. Implementation

The author simulates durations for each activity on each floor, incorporating variability to reflect real-world conditions. Python 3.11.3 on the macOS Sonoma 14.5 operating system is used to implement the computation.

a. Simulating observed data

The observed durations are generated using the true underlying distributions.

Example for Activity A (similar for other activities):
- True Beta distribution parameters: $\alpha_{true} = 4, \beta_{true} = 6$.
- Simulate $x_i \sim Beta\,(\alpha_{true}, \beta_{true})$.
- Convert to actual duration:

$$t_i = a_A + (b_A - a_A)\, x_i \tag{36}$$

b. Bayesian updating procedures and total project duration calculating

In each method, the parameters after each observation and compute the posterior estimates of activity durations are updated. Specifically, calculations for the Beta approach follow Eq. (1) to Eq. (12) and those for the Log-Normal approach follow Eq. (13) to Eq. (18).

After that, the cumulative expected total duration after each floor for all methods are computed.

4.7. Results

a. Comparison of methods

Before looking at the results, the following are expected based on previous mathematical proofs:

- Traditional PERT provides a static estimate of project duration.

- Beta-Improved Method dynamically updates estimates based on observed data, adjusting for variability.

- Log-Normal Method captures skewness in activity durations, offering potentially more accurate estimates for right-skewed data.

b. Graphical analysis

• Expected total project duration over floors

Fig. 1 depicts the expected total duration yielded by the three methods. Not surprisingly, traditional PERT would not change its estimate of the project duration at 1,020 days, due to its inability to update new data as the project progresses.

The Beta-improved model's estimates fluctuate around 1,020 days and converge to 1,030 days at the end of the project. This indicates the model's capability to incorporate observed data to refine its prediction of the project duration. The value of 1,030 days is exactly what the Log-Normal model produces at the end of the project. The difference is that the Log-Normal model starts conservatively at 1,260 days since it emphasizes the higher uncertainty present at the beginning of the project. The convergence of the results from the Beta-improved and the Log-Normal models is expected because both models have Bayesian updating mechanisms, and when uncertainty drops to minimum at the end of the project, their estimates must agree.

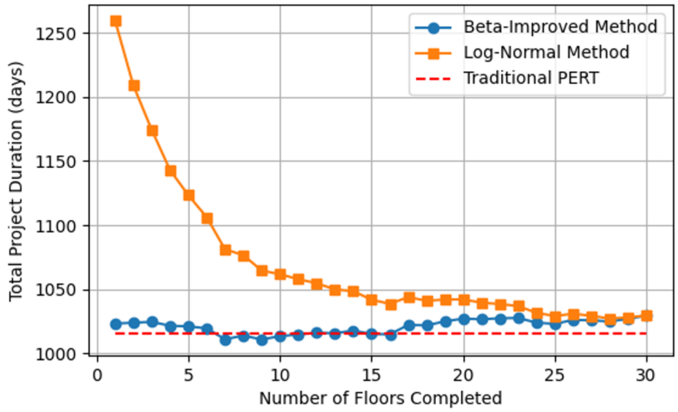


Figure 1. Expected total project duration over floors

The robustness of the proposed methods is threefold: their dynamic updating capabilities, the convergence to a reliable estimate, and uncertainty handling. The Log-Normal model shows its ability to start conservatively (due to the common right-skewed distribution of construction activities) but decrease quickly due to Bayesian prior-posterior transition.

• Variance reduction over floors

The variance reduction behaviors of the methods' estimates are depicted in Fig. 2. The traditional PERT's variance remains constant at 110 days^2, which again, is not surprising because it lacks an updating capability. The Beta-improved method, sharing the same Beta distribution with traditional PERT, reduces its variance to a barely positive value (around 40 days^2, similar to the variance of a single floor) after starting with the same values as traditional PERT. The Log-Normal method starts out very high-approximately 5,550 days^2 at the first floor – due to its consideration of the right-skewness of all 30 floors ahead. However, variance drops down to 1,500 days^2 and get stabilized around 1,400 days^2 thereafter.

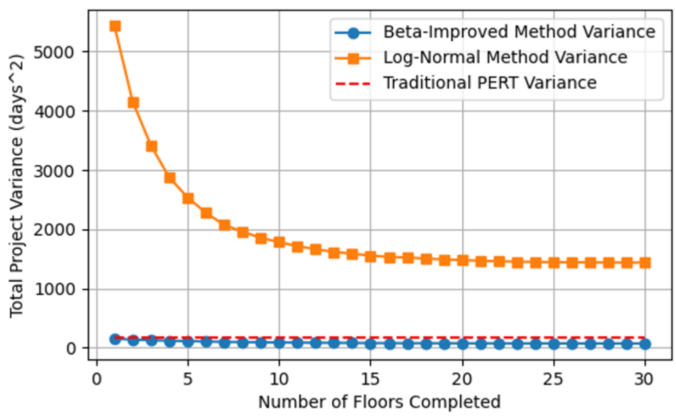


Figure 2. Variance reduction over floors

The combined observation proves the effectiveness of the proposed methods. Both Bayesian models exhibit variance reduction, though in different behaviors. While the Beta-improved can minimize its variance at the end to show its strength to reflect uncertainty reduction, the Log-Normal demonstrates its ability to deal with the right-skewed nature of construction activity durations by starting high but reducing fast.

• Probability of meeting the deadline over floors

Fig. 3 shows the probability of the project finishing on time during its course (for the case that all activities materialize at their expected duration). Since the deadline was set up at 1,050 days, the Beta-improved method maintains a steady probability of 100% that the project will finish before the deadline. However, the Log-Normal indicates a low probability of on-time completion at the beginning because the initial uncertainty is too high. Eventually, the probability increases to 75% at the 15th floor and maximizes to 82% during the construction of the last floor. This indicates the growth of confidence as uncertainty decreases. It is noteworthy that even at the end of the project of the two proposed methods converges, the probability calculated by the Log-Normal is still 82% but not 100%. Here, the right-skewed property of data in construction projects is emphasized again, representing the case that sudden risks might occur and change the whole plan upside down in the last few floors.

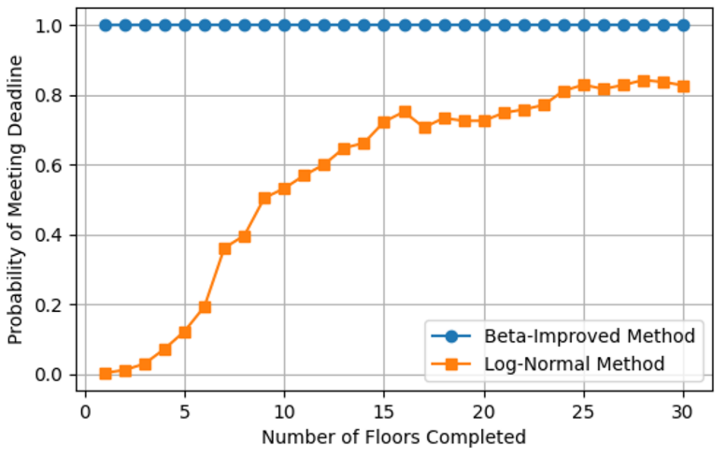


Figure 3. Probability of meeting the deadline along the project schedule

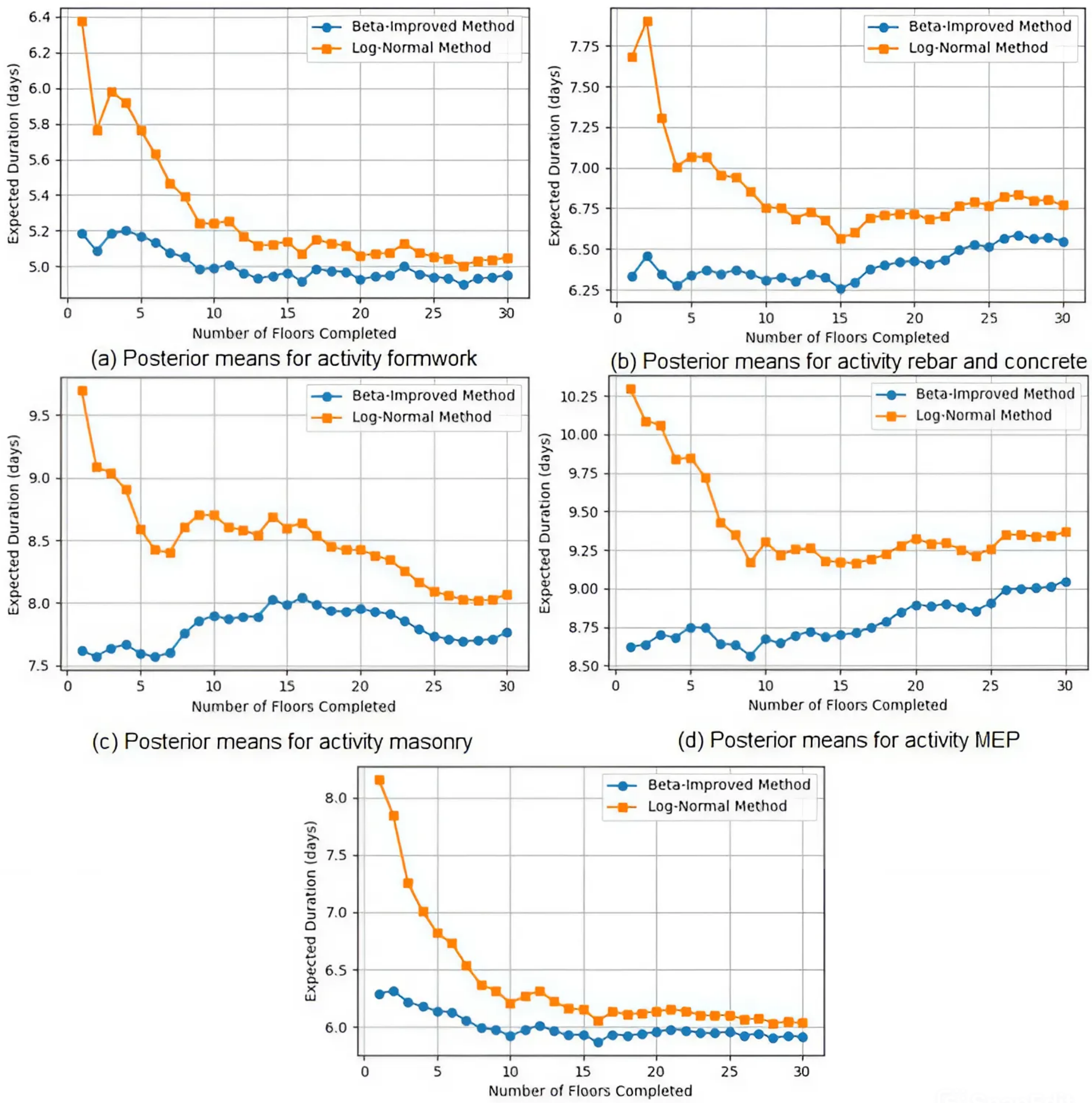
• Posterior means for each activity

Fig. 4 depicts the evolution of posterior means of five activities along the progress of the project.

Though behaving differently, the posterior means of activities of the two methods converge, with the values estimated by the Log-Normal method always greater than those of the Beta-improved method. For example, Formwork starts at 5.19 days and slowly moves down to 4.95 days according to the Beta-improved method. Meanwhile, the Log-Normal method moves in a steeper curve from 6.38 days to 5.05 days by the 30th floor.

Regarding uncertainty handling, the Log-Normal method's significant early adjustments reflect high initial uncertainty and sensitivity to new observations, highlighting its capacity to model right-skewed activity durations and potential delays. The Beta-Improved method shows more stable estimates with smaller adjustments, indicating consistent confidence in predictions and steady incorporation of observed data. This observation aligns with the comment by Schoot et al. [35], that the choice of prior distributions can significantly influence the results of Bayesian analyses.

Because both methods are based on Bayesian methods, posterior means of the activities reduce variance and converge as observed. This validates the theorems that were mathematically proved above.



(a) Posterior means for activity formwork

(b) Posterior means for activity rebar and concrete

(c) Posterior means for activity masonry

(d) Posterior means for activity MEP

(e) Posterior means for activity finishing

Figure 4. Posterior means of activities

5. Discussion on the implications for project scheduling

The Beta-improved method, sharing the Beta distribution with traditional PERT, yields consistent estimates during the schedule of the project, with minor fluctuations due to the nature of Bayesian statistics. The Log-Normal, on the other hand, begins with conservative estimates, reflecting uncertainty and right-skewed property of the distribution, but then adjusts itself as new data are updated. This swift adaptation more accurately reflects potential delays in the initial stages, which is essential for effective contingency planning.

Both methods improve risk management by decreasing variance over time, suggesting heightened confidence in the estimates. The Beta-Improved method results in a reduced final variance, thereby improving the accuracy of activity duration forecasts. The Log-Normal method shows a higher variance at the beginning but decreases as the project progresses. This conservative view allows project managers to plan for proactive mitigation strategies and may require more resources for risk management. Variance reduction translates to increased confidence in activity duration estimates as the project progresses. Early in the project, estimates are based on prior information, which may be uncertain. As actual activity durations are observed, these data points refine our estimates, reducing the posterior variance.

Benefiting from updating mechanisms, project managers can modify schedules based on actual performance, thereby optimizing timelines and minimizing idle times between activities. Accurate and updated duration estimates enhance the planning of labor, equipment, and materials, resulting in improved resource allocation and reduced costs.

These methods show their strength in application to different cases in project scheduling: the Beta-improved method might be helpful for short and simple projects and to verify schedule compliance, while the Log-Normal method can be used with long and complex projects when resources need to be monitored closely. In case sudden risks happen toward the end of a project's course, posterior variance may increase, triggering a defensive attitude of project managers, and that is when the Log-Normal method better represents the schedule. To choose which method to use, practitioners might have to make decisions based on the specific properties of each project, the experience of project managers, and iterative refinement through trial and error. However, both methods can be used in one construction project at the same time for different purposes by project practitioners to form a balanced scheduling strategy.

With both methods, the ability to dynamically update information along the course of projects is a well-known Bayesian mechanism. Continuous refinement of estimates and adjustment of plans enhance the probability of timely project completion. The consistent probability of meeting the deadline provided by the Beta-Improved method enhances confidence in the project's success. Effective scheduling minimizes the likelihood of cost overruns linked to delays, while optimal resource allocation aids in maintaining budgetary constraints. For practitioners who have experienced using traditional PERT, the migration to these proposed methods can be smooth as short workshops using synthetic data can be used to demonstrate the change in distributions and the updating mechanism of the Bayesian philosophy.

6. Conclusions

This study introduces two improvements to current PERT scheduling techniques by proposing: (i) alternative distributions with Beta with evolved parameter identification and Log-Normal distribution and (ii) A Bayesian mechanism integrated to actively update project estimates based on actual project performance.

The proposed methods are grounded in a robust mathematical foundation. The paper shows, through comprehensive mathematical formulations and demonstrations, that Bayesian updating significantly diminishes variance and approaches more precise estimates when additional data is acquired. The author formulated equations for adjusting the parameters of the Beta distribution based on observed data in the Beta-Improved technique, demonstrating that the posterior variance diminishes with increased observations and that the posterior mean approaches the true mean of activity durations. The author formulated the estimation of Log-Normal distribution parameters utilizing PERT estimations and generated Bayesian updating equations with posterior priors, demonstrating analogous convergence features.

The numerical case study of a 30-story building project demonstrated the actual application and advantages of the proposed methods. Visualized results confirm the theoretical proofs and derivations developed earlier. Improved accuracy and reliability arise from mathematically validated decreases in variance and convergence of estimates, resulting in superior planning and execution. Enhanced risk management is facilitated by the early detection of uncertainty, underpinned by mathematical proofs, which permit proactive mitigation techniques. Dynamic scheduling and resource allocation are facilitated by mathematically modeled updating capabilities, enhancing resource usage and minimizing costs.

By delivering precise and current duration estimates, supported by robust mathematical evidence, project managers may make educated decisions about milestones, deadlines, and resource allocation. The accurate modeling of project dynamics via advanced mathematics guarantees that estimates consider probable delays, crucial for establishing feasible timeframes and managing stakeholder expectations. Strategic contingency planning is improved by mathematically precise estimates, streamlining project schedules and augmenting the probability of timely completion.

One of the limitations of this study is that the models were applied in a simplified numerical case study with only a few repetitive activities. In real projects, their application might yield some unpredicted outcomes, whose cause might result from how practitioners obtain estimated parameters of activities. The assumption of interdependence among activities has not been addressed in the proposed methods. In future research, the work may be expanded so that multiple distributions are integrated into the model to treat different activities in different timeframes and in a dependent setting, and the switch is dynamic. Furthermore, resource limitations can be incorporated into the model, making it more realistic and robust. Research can include a pilot study to survey how project managers apply the models in their actual projects, thereby improving the proposed methods as a necessary management tool.

Data and code availability

The data and code used in this study are available from the corresponding author upon reasonable request.

References

[1] Adi, T. J. W., Rachmawati, F., Rizky, S. Y. (2023). Probabilistic Scheduling Based On Hybrid Bayesian Network–Program Evaluation Review Technique,. *IPTEK The Journal for Technology and Science*, 34 (2):108.

[2] Sackey, S., Kim, B.-S. (2019). Schedule Risk Analysis using a Proposed Modified Variance and Mean of the Original Program Evaluation and Review Technique Model. *KSCE Journal of Civil Engineering*, 23 (4):1484–1492.

[3] Liu, M. (2013). Program Evaluation and Review Technique (PERT) in Construction Risk Analysis. *Applied Mechanics and Materials*, 357–360:2334–2337.

[4] Ragel, L. J. B. (2021). Limitations Of Pert/Cpm In Construction Management Planning: Inputs To Mathematics In Architecture Education. *Turkish Journal of Computer and Mathematics Education (TURCOMAT)*, 12(10):5218–5223.

[5] Ashadi, R. F., Husin, A. E., Guntorojati, I. (2022). Infrastructure Construction Projects Scheduling Using Manual-Program Evaluation and Review Technique (M-PERT) Method. Case Study: Indonesian Sunda Strait Bridge. *Jurnal Teknik Sipil*, 29(2):125–132.

[6] Ballesteros-Pérez, P. (2017). M-PERT: Manual Project-Duration Estimation Technique for Teaching Scheduling Basics. *Journal of Construction Engineering and Management*, 143(9).

[7] Ballesteros-Pérez, P., Sanz-Ablanedo, E., Cerezo-Narváez, A., Lucko, G., Pastor-Fernández, A., Otero-Mateo, M., Contreras-Samper, J. P. (2020). Forecasting Accuracy of In-Progress Activity Duration and Cost Estimates. *Journal of Construction Engineering and Management*, 146(9).

[8] Habibi, F., Taghipour Birgani, O., Koppelaar, H., Radenović, S. (2018). Using fuzzy logic to improve the project time and cost estimation based on Project Evaluation and Review Technique (PERT). *Journal of Project Management*, 183–196.

[9] Kim, S. D., Hammond, R. K., Bickel, J. E. (2014). Improved Mean and Variance Estimating Formulas for PERT Analyses. *IEEE Transactions on Engineering Management*, 61(2):362–369.

[10] Premachandra, I. M. (2001). An approximation of the activity duration distribution in PERT. *Computers & Operations Research*, 28(5):443–452.

[11] Chen, C.-T., Huang, S.-F. (2007). Applying fuzzy method for measuring criticality in project network. *Information Sciences*, 177(12):2448–2458.

[12] Pesu, T., Knottenbelt, W. (2017). Optimising Hidden Stochastic PERT Networks. In *Proceedings of the 10th EAI International Conference on Performance Evaluation Methodologies and Tools*, VALUE-TOOLS, ACM.

[13] Hajdu, M., Bokor, O. (2016). Sensitivity analysis in PERT networks: Does activity duration distribution matter? *Automation in Construction*, 65:1–8.

[14] Huynh, Q.-T., Nguyen, N.-T. (2020). Probabilistic Method for Managing Common Risks in Software Project Scheduling Based on Program Evaluation Review Technique. *International Journal of Information Technology Project Management*, 11(3):77–94.

[15] García, C. B. G., Pérez, J. G., Rambaud, S. C. (2010). Proposal of a new distribution in PERT methodology. *Annals of Operations Research*, 181(1):515–538.

[16] Kirytopoulos, K. A., Leopoulos, V. N., Diamantas, V. K. (2008). PERT vs. Monte Carlo Simulation along with the suitable distribution effect. *International Journal of Project Organisation and Management*, 1 (1):24.

[17] Salas-Morera, L., Arauzo-Azofra, A., García-Hernández, L., Palomo-Romero, J. M., Ayuso-Muñoz, J. L. (2018). New Approach to the Distribution of Project Completion Time in PERT Networks. *Journal of Construction Engineering and Management*, 144(10).

[18] Khan, I. A., Bickel, J. E., Hammond, R. K. (2023). Determining the Accuracy of the Triangular and PERT Distributions. *Decision Analysis*, 20(2):151–165.

[19] Ballesteros-Pérez, P., Skitmore, M. (2016). On the distribution of bids for construction contract auctions. *Construction Management and Economics*, 35(3):106–121.

[20] van Dorp, J. R. (2020). A dependent project evaluation and review technique: A Bayesian network approach. *European Journal of Operational Research*, 280(2):689–706.

[21] Rezakhani, P. (2021). Project scheduling and performance prediction: a fuzzy-Bayesian network approach. *Engineering, Construction and Architectural Management*, 29(6):2233–2244.

[22] Wang, Z. Z., Chen, C. (2017). Fuzzy comprehensive Bayesian network-based safety risk assessment for metro construction projects. *Tunnelling and Underground Space Technology*, 70:330–342.

[23] Yu, T., Man, Q., Wang, Y., Shen, G. Q., Hong, J., Zhang, J., Zhong, J. (2019). Evaluating different stakeholder impacts on the occurrence of quality defects in offsite construction projects: A Bayesian-network-based model. *Journal of Cleaner Production*, 241:118390.

[24] Wu, X., Liu, H., Zhang, L., Skibniewski, M. J., Deng, Q., Teng, J. (2015). A dynamic Bayesian network based approach to safety decision support in tunnel construction. *Reliability Engineering & System Safety*, 134:157–168.

[25] Cai, B., Kong, X., Liu, Y., Lin, J., Yuan, X., Xu, H., Ji, R. (2019). Application of Bayesian Networks in Reliability Evaluation. *IEEE Transactions on Industrial Informatics*, 15(4):2146–2157.

[26] Gondia, A., Siam, A., El-Dakhakhni, W., Nassar, A. H. (2020). Machine Learning Algorithms for Construction Projects Delay Risk Prediction. *Journal of Construction Engineering and Management*, 146 (1).

[27] Cho, S. (2009). A linear Bayesian stochastic approximation to update project duration estimates. *European Journal of Operational Research*, 196(2):585–593.

[28] Price, L. F., Drovandi, C. C., Lee, A., Nott, D. J. (2017). Bayesian Synthetic Likelihood. *Journal of Computational and Graphical Statistics*, 27(1):1–11.

[29] Müller, P., Quintana, F. A. (2004). Nonparametric Bayesian Data Analysis. *Statistical Science*, 19(1).

[30] Hendradewa, A. P. (2019). Schedule Risk Analysis by Different Phases of Construction Project Using CPM-PERT and Monte-Carlo Simulation. *IOP Conference Series: Materials Science and Engineering*, 528(1):012035.

[31] Ash, R. C., Pittman, P. H. (2008). Towards holistic project scheduling using critical chain methodology enhanced with PERT buffering. *International Journal of Project Organisation and Management*, 1(2): 185.

[32] Kim, B.-C. (2016). Probabilistic Evaluation of Cost Performance Stability in Earned Value Management. *Journal of Management in Engineering*, 32(1).

[33] Ba'Its, H. A., Puspita, I. A., Bay, A. F. (2020). Combination of program evaluation and review technique (PERT) and critical path method (CPM) for project schedule development. *International Journal of Integrated Engineering*, 12(3):68–75.

[34] Levner, E., Kats, V., Yan, P., Che, A. (2024). Fast Algorithm for High-Throughput Screening Scheduling Based on the PERT/CPM Project Management Technique. *Algorithms*, 17(3):127.

[35] van de Schoot, R., Depaoli, S., King, R., Kramer, B., Märtens, K., Tadesse, M. G., Vannucci, M., Gelman, A., Veen, D., Willemsen, J., Yau, C. (2021). Bayesian statistics and modelling. *Nature Reviews Methods Primers*, 1(1).