FREE VIBRATION ANALYSIS OF FUNCTIONALLY GRADED TIMOSHENKO BEAM USING DYNAMIC STIFFNESS METHOD

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***Summary**: In this paper, free vibration of a functionally graded material (FGM) beam is investigated based on the Timoshenko beam theory and dynamic stiffness method. Material properties vary continuously throughout the thickness direction according to the volume fraction constituent defined by power law function. First, a consistent theory of vibration is established for FGM Timoshenko beam taking into account the actual position of neutral axis that is a useful tool for analysis of coupled vibration in the beam. Then, frequency equation obtained provides an efficient method for free vibration analysis of FGM Timoshenko beam. The theoretical development has been illustrated and validated by numerical examples.*

***Keywords**: FGM; Timoshenko beam; natural frequency; dynamic stiffness method.*

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1. Introduction

Functionally graded materials (FGMs) are inhomogeneous composites characterized by smooth and continuous variations in both compositional profile and material properties. FGMs are widely used in many scientific and engineering fields, such as aerospace, automobile, electronics, optics, chemistry, biomedical engineering, nuclear engineering and mechanical engineering. FGMs have been proved to be advanced materials by their advantaged properties compared to the laminate composites and by the wide application in the high-tech industries.

Various methods have been developed for free vibration analysis of FGM beam. The analytical methods have been shown to be most accurate and efficient for dynamic analysis of FGM beam-like structures [1-5]. A number of authors have developed approximate methods such as the Finite Element Method (FEM) [6-7], Rayleigh-Ritz method [8], combined Fourier series - Galerkin method [9],...

As the FEM is established by using frequency-independent polynomial shape function, the FEM cannot capture all necessary high frequencies [10]. An alternative approach improved the solution accuracy is to use shape functions that depended on vibration frequency. This elegant concept led to the so-called Dynamic Stiffness Method (DSM) [11]. The DSM uses the frequency dependent shape functions obtained from the exact solution of the governing differential equations of motion in free vibration. The method provides exact results for all natural frequencies and mode shapes without making any approximation on the way. The DSM is recognized as the most accurate method in free vibration analysis, significantly superior to traditional FEM and other approximate methods. By combining the advantaged features of the DSM with those of the spectral analysis method, the authors of Ref. [12] introduced the fundamental concept of the Spectral Element Method (SEM) for the first time in his 1978 work [13].

Apparently, there has been very little effort to solve the free vibration problem of FGM beams using the DSM. The present research is based on earlier research [14-17] using DSM, but includes many additional features with wide-ranging results for free vibration of FGM beams. It focuses on applying the DSM and investigating the free vibration of FGM beams for different boundary conditions.

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While most of the aforementioned studies are based on the assumption that the neutral axis of FGM beam coincides with the central one, the authors of Ref. [18] have determined exact position of the neutral axis and investigated its effect on natural frequencies of FGM Euler-Bernoulli beams.

In this paper, the exact method developed on the DSM for dynamical analysis of FGM Timoshenko beam is presented. Governing equations for vibration of FGM Timoshenko beam are conducted on the base of power law distribution of FGM taking into account the actual position of neutral axis. The frequency equations obtained provide a simple approach to analyze free vibration of FGM beam for different boundary conditions. Usefulness of the theoretical development is validated by numerical results.

2. Governing equations

Consider a FGM beam of length L, cross section area A=b×h. It is assumed that the material properties of FGM beam vary along the thickness direction by a power law distribution [16].

$$\begin{Bmatrix} E(z) \\ G(z) \\ \rho(z) \end{Bmatrix} = \begin{Bmatrix} E_b \\ G_b \\ \rho_b \end{Bmatrix} + \begin{Bmatrix} E_t - E_b \\ G_t - G_b \\ \rho_t - \rho_b \end{Bmatrix} \left(\frac{z}{h} + \frac{1}{2} \right)^n, -h/2 \le z \le h/2 \tag{1}$$

where E, G and ρ stand for Young's, shear modulus and mass density and n is power law exponent, t and b denote the top and bottom materials respectively; z is co-ordinate of the point from the mid plane axis at high h/2. It is assumed that the displacement fields in the cross section at x are

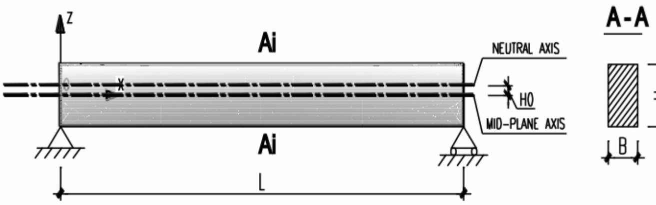

Figure 1. FGM beam

$$u(x,z,t) = u_0(x,t) - (z - h_0)\theta(x,t)$$
$$w(x,z,t) = w_0(x,t) \tag{2}$$

Where u_0, v_0 are the displacements of neutral axis located at the high h_0 from the central axis; θ is the rotation of the cross section.

Therefore, the normal strain ε_x and shear strain γ_{xz} can be expressed as follows

$$\varepsilon_x = \partial u_0 / \partial x - (z - h_0)\partial\theta / \partial x; \gamma_{xz} = \partial w_0 / \partial x - \theta \tag{3}$$

And the normal stress σ_x and shear stress τ_{xz} are given by linear elastic constitutive law as

$$\sigma_x = E(z)\varepsilon_x; \tau_{xz} = \psi G(z)\gamma_{xz} \tag{4}$$

Suppose that the beam is subjected to distributed loads: axial n(x,t), flexural p(x,t) and bending moment m(x,t). Based on the Hamilton's principle, the equations of motion can be established in the time domain [7,17] as follows

$$\begin{gathered} \delta u_0 : (I_{11}\ddot{u}_0 - A_{11}u_0'') - (I_{12}\ddot{\theta} - A_{12}\theta'') = n(x,t) \\ \delta\theta : (I_{12}\ddot{u}_0 - A_{12}u_0'') - (I_{22}\ddot{\theta} - A_{22}\theta'') + A_{33}(w_0' - \theta) = m(x,t) \\ \delta w_0 : I_{11}\ddot{w}_0 - A_{33}(w_0'' - \theta') = p(x,t) \end{gathered} \tag{5}$$

where

$$\begin{gathered} (A_{11}, A_{12}, A_{22}) = \int_A E(z)(1, z - h_0, (z - h_0)^2)dA;\ A_{33} = \psi \int_A G(z)dA \\ (I_{11}, I_{12}, I_{22}) = \int_A \rho(z)(1, z - h_0, (z - h_0)^2)dA. \end{gathered} \tag{6}$$

Using the power law distribution (1) for FGM material, the above constants (6) can be calculated as

$$\begin{gathered} A_{11} = bh\frac{(E_t + nE_b)}{1+n};\ I_{11} = bh\frac{(\rho_t + n\rho_b)}{1+n};\ A_{12} = bh^2\left[\frac{2E_t + nE_b}{2(2+n)} - \frac{E_t + nE_b}{(1+n)}\alpha\right] \\ I_{12} = bh^2\left[\frac{2\rho_t + n\rho_b}{2(2+n)} - \frac{\rho_t + n\rho_b}{(1+n)}\alpha\right];\ A_{22} = bh^3\left[\frac{3E_t + nE_b}{3(3+n)} - \frac{2E_t + nE_b}{(2+n)}\alpha + \frac{E_t + nE_b}{(1+n)}\alpha^2\right] \\ I_{22} = bh^3\left[\frac{3\rho_t + n\rho_b}{3(3+n)} - \frac{2\rho_t + n\rho_b}{(2+n)}\alpha + \frac{\rho_t + n\rho_b}{(1+n)}\alpha^2\right];\ A_{33} = bh\psi\frac{(G_t + nG_b)}{1+n} \end{gathered} \tag{7}$$

$$\alpha = 1/2 + h_0/h$$

Actual position of the neutral axis h_0 (Fig. 1) is defined as $A_{12}=0$ [18], one gets

$$h_0 = \frac{n(r_E - 1)h}{2(n+2)(n+r_E)}, r_E = \frac{E_t}{E_b}. \tag{8}$$

Introducing the amplitudes of axial displacement, rotation, transverse displacement and loads

$$\{U,\Theta,W\} = \int_{-\infty}^{\infty}\{u_0(x,t),\theta(x,t),w_0(x,t)\}e^{-i\omega t}dt$$

$$\{\overline{N}(x,\omega),\overline{M}(x,\omega),\overline{P}(x,\omega)\} = \int_{-\infty}^{\infty}\{n(x,t),m(x,t),p(x,t)\}e^{-i\omega t}dt \tag{9}$$

Equations (5) become

$$\begin{gathered}(\omega^2 I_{11}U + A_{11}U'') - \omega^2 I_{12}\Theta - A_{12}\Theta'' = -\overline{N}\\ (\omega^2 I_{22}\Theta + A_{22}\Theta'') - \omega^2 I_{12}U - A_{12}U'' + A_{33}(W' - \Theta) = -\overline{M}\\ \omega^2 I_{11}W + A_{33}(W'' - \Theta') = -\overline{P}\end{gathered} \tag{10}$$

Using the following matrix and vector notations

$$\mathbf{A} = \begin{bmatrix} A_{11} & -A_{12} & 0\\ -A_{12} & A_{22} & 0\\ 0 & 0 & A_{33}\end{bmatrix}; \mathbf{\Pi} = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & A_{33}\\ 0 & -A_{33} & 0\end{bmatrix}; \mathbf{C}(\omega) = \begin{bmatrix} \omega^2 I_{11} & -\omega^2 I_{12} & 0\\ -\omega^2 I_{12} & \omega^2 I_{22} - A_{33} & 0\\ 0 & 0 & \omega^2 I_{11}\end{bmatrix} \tag{11}$$

$$\mathbf{z} = \{U,\Theta,W\}^T, \mathbf{q} = \{\overline{N},\overline{M},P\}^T \tag{12}$$

Equations (10) are rewritten in form

$$\mathbf{A}\mathbf{z}'' + \mathbf{\Pi}\mathbf{z}' + \mathbf{C}\mathbf{z} = -\mathbf{q} \tag{13}$$

3. Frequency equation

Equation of motion (13) in the case of free vibration is

$$\mathbf{A}\mathbf{z}'' + \mathbf{\Pi}\mathbf{z}' + \mathbf{C}\mathbf{z} = 0 \tag{14}$$

A continuous solution of Eq. (14) can be sought in the form $\mathbf{z}_0 = \mathbf{d}e^{\lambda x}$ that yields

$$[\lambda^2\mathbf{A} + \lambda\mathbf{\Pi} + \mathbf{C}]\mathbf{d} = 0 \tag{15}$$

This equation would have nontrivial solution with respect to constant vector d under the condition

$$\det[\lambda^2\mathbf{A} + \lambda\mathbf{\Pi} + \mathbf{C}] = 0 \tag{16}$$

It is in fact a cubic algebraic equation with respect to $\eta = \lambda^2$ that can be elementarily solved and gives three roots η_1, η_3, η_3. Therefore, solutions of Eq. (16) are

$$\lambda_{1,4} = \pm k_1; \lambda_{2,5} = \pm k_2; \lambda_{3,6} = \pm k_3\ ; k_j = \sqrt{\eta_j}, j = 1,2,3 \tag{17}$$

Now, general continuous solution of Eq. (14) can be represented as

$$\mathbf{z}_0(x,\omega) = \sum_{j=1}^{6}\mathbf{d}_j e^{\lambda_j x} \tag{18}$$

Or in the matrix form

$$\mathbf{z}_0 = \begin{bmatrix} d_{11} & d_{12} & \dots & d_{16}\\ d_{21} & d_{22} & \dots & d_{26}\\ d_{31} & d_{32} & \dots & d_{36}\end{bmatrix}\cdot\begin{Bmatrix} e^{\lambda_1 x}\\ \vdots\\ e^{\lambda_6 x}\end{Bmatrix}$$

Taking into account the second and last equations in (10) one gets

$$\begin{bmatrix} d_{11} & d_{12} & \dots & d_{16}\\ d_{21} & d_{22} & \dots & d_{26}\\ d_{31} & d_{32} & \dots & d_{36}\end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} & \dots & d_{16}\\ \alpha_1 d_{11} & \alpha_2 d_{12} & \dots & \alpha_6 d_{16}\\ \beta_1 d_{11} & \beta_2 d_{12} & \dots & \beta_6 d_{16}\end{bmatrix} \tag{19}$$

where

$$\alpha_j = \frac{\omega^2 I_{11} + \lambda_j^2 A_{11}}{\omega^2 I_{12} + A_{12}\lambda_j^2}; \beta_j = \frac{\lambda_j A_{33}\left(\omega^2 I_{11} + \lambda_j^2 A_{11}\right)}{\left(\omega^2 I_{11} + \lambda_j^2 A_{33}\right)\left(\omega^2 I_{12} + A_{12}\lambda_j^2\right)}; j = 1,2,\dots,6 \tag{20}$$

Using the notations introduced in (17) it is easy to verify that

$\alpha_4 = \alpha_1; \alpha_5 = \alpha_2; \alpha_6 = \alpha_3; \beta_4 = -\beta_1; \beta_5 = -\beta_2; \beta_6 = -\beta_3$

And expression (18) can be now rewritten in the form

$$\mathbf{z}_0(x,\omega) = \mathbf{G}(x,\omega)\mathbf{C} \tag{21}$$

With $\mathbf{C} = (C_1,...,C_6)^T = (d_{11},...,d_{16})^T$ and

$$\mathbf{G}(x,\omega) = [\mathbf{G}_1(x,\omega)\ \ \mathbf{G}_2(x,\omega)]$$

$$\mathbf{G}_1(x,\omega) = \begin{bmatrix} e^{k_1x} & e^{k_2x} & e^{k_3x} \\ \alpha_1 e^{k_1x} & \alpha_2 e^{k_2x} & \alpha_3 e^{k_3x} \\ \beta_1 e^{k_1x} & \beta_2 e^{k_2x} & \beta_3 e^{k_3x} \end{bmatrix}^T; \mathbf{G}_2(x,\omega) = \begin{bmatrix} e^{-k_1x} & e^{-k_2x} & e^{-k_3x} \\ \alpha_1 e^{-k_1x} & \alpha_2 e^{-k_2x} & \alpha_3 e^{-k_3x} \\ -\beta_1 e^{-k_1x} & -\beta_2 e^{-k_2x} & -\beta_3 e^{-k_3x} \end{bmatrix} \tag{22}$$

Decomposing the constant vector $\mathbf{C} = \{\mathbf{C}_1, \mathbf{C}_2\}^T$ with $\mathbf{C}_1 = \{\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3\}^T$, $\mathbf{C}_2 = \{\mathbf{C}_4, \mathbf{C}_5, \mathbf{C}_6\}^T$ and letting nodal displacements at the beam ends be denoted by

$$\mathbf{D}_0 = \{U_0, \Theta_0, W_0\}^T = \mathbf{z}_0(0,\omega); \mathbf{D}_L = \{C_4, C_5, C_6\}^T = \mathbf{z}_0(L,\omega) \tag{23}$$

One gets

$$\mathbf{G}_{01}\mathbf{C}_1 + \mathbf{G}_{02}\mathbf{C}_2 = \mathbf{D}_0, \mathbf{G}_{L1}\mathbf{C}_1 + \mathbf{G}_{L2}\mathbf{C}_2 = \mathbf{D}_L \tag{24}$$

where

$$\mathbf{G}_{01}(\omega) = \mathbf{G}_1(0,\omega); \mathbf{G}_{02}(\omega) = \mathbf{G}_2(0,\omega); \mathbf{G}_{L1}(\omega) = \mathbf{G}_1(L,\omega); \mathbf{G}_{L1}(\omega) = \mathbf{G}_2(L,\omega) \tag{25}$$

Obviously, Eq. (24) allows one to find

$$\mathbf{C}_1 = \mathbf{H}_{10}\mathbf{D}_0 + \mathbf{H}_{1L}\mathbf{D}_L; \mathbf{C}_2 = \mathbf{H}_{20}\mathbf{D}_0 + \mathbf{H}_{2L}\mathbf{D}_L$$

where

$$\mathbf{H}_{10} = \frac{G_{L2}}{G_{01}G_{L2} - G_{L1}G_{02}}; \mathbf{H}_{1L} = \frac{-G_{02}}{G_{01}G_{L2} - G_{L1}G_{02}}; \mathbf{H}_{20} = \frac{-G_{L1}}{G_{01}G_{L2} - G_{L1}G_{02}}; \mathbf{H}_{2L} = \frac{G_{01}}{G_{01}G_{L2} - G_{L1}G_{02}}$$

So that the solution $\mathbf{z}_0(x)$ can be expressed as

$$\mathbf{z}_0(x,\omega) = \mathbf{\Gamma}_0(x,\omega)\mathbf{D}_0 + \mathbf{\Gamma}_L(x,\omega)\mathbf{D}_L \tag{26}$$

With

$$\mathbf{\Gamma}_0(x,\omega) = \mathbf{G}_1(x,\omega)\mathbf{H}_{10} + \mathbf{G}_2(x,\omega)\mathbf{H}_{20};\ \mathbf{\Gamma}_L(x,\omega) = \mathbf{G}_1(x,\omega)\mathbf{H}_{1L} + \mathbf{G}_2(x,\omega)\mathbf{H}_{2L} \tag{27}$$

Furthermore, suppose that vector of internal forces is defined by

$$\begin{aligned} \mathbf{P}(x,\omega) &= \{N, M, Q\}^T = \Im\{\mathbf{z}_0(x,\omega)\} = \mathbf{F}_0(x,\omega)\mathbf{D}_0 + \mathbf{F}_L(x,\omega)\mathbf{D}_L; \\ \mathbf{F}_0(x,\omega) &= \Im\{\mathbf{\Gamma}_0(x,\omega)\}; \mathbf{F}_L(x,\omega) = \Im\{\mathbf{\Gamma}_L(x,\omega)\}. \end{aligned} \tag{28}$$

Therefore, nodal force vectors would be calculated as

$$\begin{aligned} \mathbf{P}_0 &= \{N_0, M_0, Q_0\}^T = \mathbf{F}_0(0,\omega)\mathbf{D}_0 + \mathbf{F}_L(0,\omega)\mathbf{D}_L; \\ \mathbf{P}_L &= \{N_L, M_L, Q_L\}^T = \mathbf{F}_0(L,\omega)\mathbf{D}_0 + \mathbf{F}_L(L,\omega)\mathbf{D}_L \end{aligned} \tag{29}$$

And can be rewritten in the form

$$\begin{Bmatrix} \mathbf{P}_0 \\ \mathbf{P}_L \end{Bmatrix} = \begin{bmatrix} \mathbf{F}_0(0,\omega) & \mathbf{F}_L(0,\omega) \\ \mathbf{F}_0(L,\omega) & \mathbf{F}_L(L,\omega) \end{bmatrix} \cdot \begin{Bmatrix} \mathbf{D}_0 \\ \mathbf{D}_L \end{Bmatrix} \tag{30}$$

The matrix

$$[\mathbf{K}(\omega)] = \begin{bmatrix} \mathbf{F}_0(0,\omega) & \mathbf{F}_L(0,\omega) \\ \mathbf{F}_0(L,\omega) & \mathbf{F}_L(L,\omega) \end{bmatrix} = [k_{ij}(\omega), i, j = 1,2,...,6] \tag{31}$$

is usually called dynamic stiffness matrix for the FGM beam element. The Eq. (30) can be used for establishing frequency equation for beam with different boundary conditions as follows:

a) In the case of simply supported (SS) where boundary conditions are

$$U_0 = W_0 = W_L = M_0 = N_L = M_L = 0 \tag{32}$$

Eq. (30) yields

$$k_{12}\Theta_0 + k_{14}U_L + k_{15}\Theta_L = 0; k_{22}\Theta_0 + k_{24}U_L + k_{25}\Theta_L = 0; k_{52}\Theta_0 + k_{54}U_L + k_{55}\Theta_L = 0$$

Hence frequency equation for SS beam gets the form

$$f_{SS}(\omega) = \det\begin{bmatrix} k_{12} & k_{14} & k_{15} \\ k_{22} & k_{24} & k_{25} \\ k_{52} & k_{54} & k_{55} \end{bmatrix} = 0 \quad (33)$$

b) Similarly, for cantilever (CF) beam where

$$U_0 = W_0 = \Theta_0 = N_L = M_L = Q_L = 0 \quad (34)$$

Frequency equation is

$$f_{CF}(\omega) = \det\begin{bmatrix} k_{44} & k_{45} & k_{46} \\ k_{54} & k_{55} & k_{56} \\ k_{64} & k_{65} & k_{66} \end{bmatrix} = 0 \quad (35)$$

c) Frequency equation for clamped end (CC) beam is obtained as special case as follows. Since nodal displacements equal to zero, equations (24) yield

$$\mathbf{G}_{01}\mathbf{C}_1 + \mathbf{G}_{02}\mathbf{C}_2 = 0, \mathbf{G}_{L1}\mathbf{C}_1 + \mathbf{G}_{L2}\mathbf{C}_2 = 0 \quad (36)$$

So that it must be satisfied condition

$$f_{CC}(\omega) = \det\begin{bmatrix} \mathbf{G}_{01}(\omega) & \mathbf{G}_{02}(\omega) \\ \mathbf{G}_{L1}(\omega) & \mathbf{G}_{L1}(\omega) \end{bmatrix} = 0 \quad (37)$$

Eq. (37) is desired frequency equation for CC beam.

Thus, roots of the frequency equations established above provide natural frequencies ω_i of FGM beam. In order to compare the results, the non-dimensional natural frequency λ_i is calculated as

$$\lambda_i = \frac{\omega_i L^2}{h}\sqrt{\frac{\rho_b}{E_b}} \quad (38)$$

4. Numerical results and discussion

4.1 Comparison of natural frequencies of homogeneous beam

Consider a homogeneous SS beam made of pure Aluminum (Al) with following material properties: E=70GPa, ρ=2700kg/m^3, μ=0.3, and cross sectional dimensions: b=0.1m, h=0,1m [16]. In this degenerated case E_t=E_b=E, the power law exponent in the equation (1) is n=0.

The comparison of non-dimensional natural frequencies i for homogeneous Timoshenko SS beam for three values of L/h is presented in Table 1 where present results are compared to analytical solution [19] using Timoshenko beam theory (TBT) and those given Ref.[16] (using TBT). A good agreement between them is observed.

Table 1. *Comparison of non-dimensional natural frequencies i for homogeneous SS beam*

Freq.	*L/h* =10			*L/h* =30			*L/h* =100		
λ_i	Present	Ref. [19]	Ref.[16]	Present	Ref. [19]	Ref. [16]	Present	Ref. [19]	Ref. [16]
1	2.8023	2.8132	2.8023	2.8436	2.8451	2.8439	2.8451	2.8487	2.8496
2	10.7087	10.8530	-	11.3111	11.3320	-	11.3748	11.3906	-
3	22.5612	23.1117	-	25.2191	25.3192	-	25.5717	25.6124	-
4	37.1424	38.3823		44.2819	44.5802		45.4071	45.4925	
5	53.4963	55.5937		68.1368	68.8173		70.8414	71.0004	

4.2 The influence of the position of neutral axis

Based on the equation (8), the effect of elasticity ratio E_t/E_b and material power law exponent n on the position of neutral axis is presented in Fig. 2. The neutral axis is shifted from mid-plane as the power law exponent n increases from n=3 to n=5 but decrease after that when n is higher. They coincide together when n=∞.

Fig. 3 presents the deviation between the calculated 1st non-dimensional frequency (λ_1) using neutral axis (NA) and using mid-plane axis (MA) corresponding to the different elasticity ratios E_t/E_b and power law exponent n. The FGM SS beam has the following material properties [17]: E_b=210GPa, ρ_b=7800kg/m^3, μ_b=0.31 and geometric dimensions: b=0.1m, h=0.1m, L/h=10. The figure shows that, as the elasticity ratio increases, difference between fundamental dimensionless frequencies calculated using NA and MA increases. Also, as the power law exponent n increases from 0 to 4 the error increases, after that, the increasing in graduation tends to reduce error. When n is smaller than 1 (n<1), the deviation of frequency λ_1 using NA and MA is small and lightly dependent of elasticity ratio E_t/E_b.

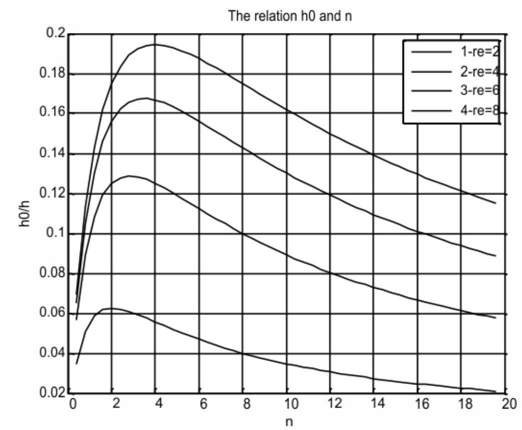

Figure 2. *Effect of elasticity ratio E_t/E_b and power law exponent n on the position of the neutral axis*

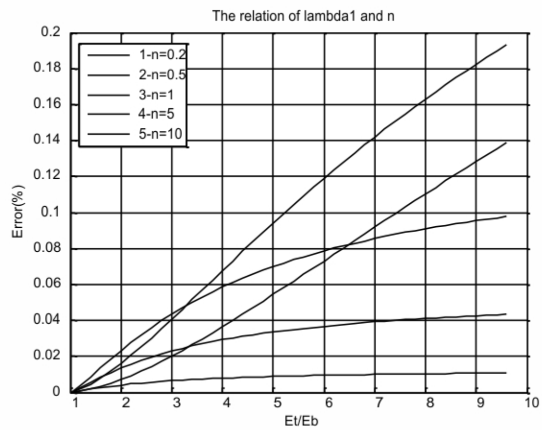

Figure 3. *Deviation between the calculated 1st non-dimensional frequency (λ_1) using NA and MA with the different ratios E_t/E_b and n*

4.3 The influence of boundary conditions

A FGM beam with the following cross sectional dimensions: b=0.1m, h=0.1m; material properties: Alumina (Al_2O_3) in the top surface: E_t=390GPa, ρ_t=3960kg/m^3, μ_t=0.25; Steel in the bottom surface: E_b=210GPa, ρ_b=7800kg/m^3, μ_b=0.31 [16].

Table 2 shows the non-dimensional natural frequencies λ_i calculated by present study and by Su, Banerjee (S&B) [16] for FGM beam using NA, MA with the different ratios L/h, power law exponent n=10 and boundary conditions: SS, CC, CF. It is observed that the discrepancy between the results from this study with those in [16] is quite small.

Tables 3-5 present the non-dimensional natural frequencies λ_i calculated by the study and by S&B [16] for the beam using MA with the different ratios L/h, power law exponent n and boundary conditions: SS, CC, CF. It is seen that the discrepancy between the sets of results from present study to those given by S&B is quite small.

Table 2. *Comparison of non-dimensional natural frequencies λ_i of FGM beam using NA and MA*

		Power law exponent (*n*) – n=10								
		SS-FGM beam			CC-FGM beam			CF-FGM beam		
L/h	Fq	S&B[16]	(NA)	(MA)	S&B[16]	(NA)	(MA)	S&B[16]	(NA)	(MA)
5	1	2.9513	2.9369	2.9375	5.6680	5.6484	5.6484	1.0867	1.0813	1.0814
	2	10.176	10.1299	10.1282	12.896	12.8530	12.8530	5.8159	5.7888	5.7919
	3	16.686	16.6648	16.6796	16.686	16.6727	16.6727	8.3430	8.3503	8.3526
	4	19.331	19.2538	19.2532	21.428	21.3620	21.3620	13.776	13.7143	13.7172
	5	29.194	29.0943	29.0958	30.559	30.4735	30.4735	22.783	22.6916	22.6952

10	1	3.0959	3.0805	3.0806	6.6638	6.6339	6.6339	1.1130	1.1074	1.074
	2	11.805	11.7476	11.7474	17.014	16.9432	16.9432	6.6562	6.6236	6.6247
	3	24.799	24.6834	24.6839	30.647	30.5295	30.5295	16.686	16.6779	16.6779
	4	33.371	33.3619	33.3641	33.371	33.3620	33.3620	17.459	17.3766	17.3791
	5	40.700	40.5218	40.5287	46.401	46.2376	46.2376	31.575	31.4296	31.4518
20	1	3.1363	3.1206	3.1206	7.0116	6.9776	6.9776	1.1199	1.1143	1.1143
	2	12.383	12.3218	12.3218	18.892	18.8040	18.8040	6.9324	6.8979	6.8982
	3	27.291	27.1584	27.1587	35.987	35.8250	35.8250	19.394	18.9503	18.9512
	4	47.214	46.9903	46.9890	57.504	57.2555	57.2555	33.372	33.3679	33.3690
	5	66.743	66.7400	66.7451	66.743	66.7407	66.7407	36.345	36.1714	36.1745
30	1	3.1440	3.1282	3.1282	7.0825	7.0476	7.0476	1.1212	1.1156	1.1156
	2	12.502	12.4399	12.4399	19.319	19.2257	19.2257	6.9876	6.9528	6.9529
	3	27.861	27.7241	27.7243	37.355	37.1805	37.1805	19.394	19.2986	19.2991
	4	48.888	48.6527	48.6525	60.718	60.4428	60.4428	37.532	37.3503	37.3518
	5	75.163	74.8060	74.8108	88.946	88.5575	88.5575	50.058	50.0599	50.0612

Table 3. *Comparison of non-dimensional natural frequencies λ_i for SS beam*

L/h	Fq	n=0.1		n=0.5		n=1		n=5		n=10	
		Present	S&B[16]	Present	S&B[16]	Present	S&B[16]	Present	S&B[16]	Present	S&B[16]
5	1	4.7807	4.784	4.0256	4.0590	3.6334	3.6890	3.0954	3.1088	2.9369	2.9513
	2	16.6399	16.652	14.0095	14.128	12.6270	12.818	10.6124	10.721	10.1299	10.176
	3	28.1772	28.189	23.9367	24.022	21.5052	21.621	17.4741	17.526	16.6648	16.686
	4	31.9007	31.924	26.8653	27.085	24.1855	24.531	20.1835	20.366	19.2538	19.331
	5	48.5456	48.579	40.9073	41.215	36.7950	37.269	30.5188	30.757	29.0943	29.194
10	1	4.9977	5.001	4.2086	4.2432	3.8004	3.8586	3.2251	3.2608	3.0805	3.0959
	2	19.1228	19.136	16.1021	16.235	14.5331	14.755	12.3013	12.434	11.7476	11.805
	3	40.3570	40.385	33.9801	34.261	30.6491	31.110	25.8533	26.122	24.6834	24.799
	4	56.3731	56.379	48.0050	48.044	43.1884	43.242	35.0281	35.052	33.3619	33.371
	5	66.5611	66.608	56.0479	56.502	50.5213	51.256	42.4555	42.873	40.5218	40.700
20	1	5.0579	5.0613	4.2594	4.2943	3.8468	3.9058	3.2669	3.3032	3.1206	3.1363
	2	19.9907	20.004	16.8343	16.972	15.2015	15.433	12.9004	13.042	12.3218	12.383
	3	44.1262	44.156	37.1577	37.460	33.5463	34.052	28.4358	28.743	27.1584	27.291
	4	76.4911	76.542	64.4085	64.928	58.1323	58.997	49.2052	49.725	46.9903	47.214
	5	112.7558	112.76	96.0780	96.090	86.4691	86.486	70.0965	70.104	66.7400	66.743
30	1	5.0694	5.0727	4.2691	4.3041	3.8557	3.9147	3.2749	3.3113	3.1282	3.1440
	2	20.1682	20.181	16.9841	17.123	15.3384	15.572	13.0235	13.167	12.4399	12.502
	3	44.9790	45.009	37.8773	38.184	34.2035	34.719	29.0258	29.342	27.7241	27.861
	4	79.0067	79.058	66.5306	67.063	60.0693	60.962	50.9395	51.484	48.6527	48.888
	5	121.6210	121.70	102.4129	103.22	92.451	93.801	78.3316	79.149	74.806	75.163

Table 4. *Comparison of non-dimensional natural frequencies λ_i for CC beam*

L/h	Fq	n=0.1		n=0.5		n=1		n=5		n=10	
		Present	S&B[16]	Present	S&B[16]	Present	S&B[16]	Present	S&B[16]	Present	S&B[16]
5	1	9.3334	9.3380	7.8761	7.9241	7.0980	7.1772	5.9244	5.9699	5.6484	5.6680
	2	21.4415	21.455	18.0842	18.206	16.2684	16.459	13.4826	13.585	12.8530	12.896
	3	28.1816	28.189	23.9661	24.022	21.5450	21.621	17.4937	17.526	16.6727	16.686
	4	35.8024	35.825	30.1987	30.399	27.1480	27.456	22.4161	22.573	21.3620	21.428
	5	51.2185	51.248	43.2147	43.484	38.8398	39.251	31.9895	32.193	30.4735	30.559

10	1	10.8205	10.827	9.1182	9.1864	8.2292	8.3437	6.9493	7.0184	6.6339	6.6638
	2	27.7924	27.809	23.4254	23.594	21.1256	21.404	17.7560	17.909	16.9432	17.014
	3	50.3343	50.364	42.4315	42.727	38.2389	38.721	32.0050	32.279	30.5295	30.647
	4	56.3731	56.379	48.0050	48.044	43.1884	43.242	35.0282	35.052	33.3620	33.371
	5	76.5667	76.611	64.5588	64.989	58.1469	58.840	48.4886	48.873	46.2376	46.401
20	1	11.3264	11.334	9.5401	9.6159	8.6146	8.7425	7.3059	7.3844	6.9776	7.0116
	2	30.5823	30.602	25.7612	25.961	23.2565	23.593	19.6918	19.896	18.8040	18.892
	3	58.3934	58.430	49.1912	49.565	44.3954	45.019	37.5224	37.898	35.8250	35.987
	4	93.5495	93.607	78.8087	79.395	71.1008	72.072	59.9769	60.554	57.2555	57.504
	5	112.7561	112.76	96.0796	96.090	86.4714	86.486	70.0980	70.104	66.7407	66.743
30	1	11.4287	11.326	9.6253	9.7027	8.6926	8.8232	7.3786	7.4591	7.0476	7.0825
	2	31.2055	31.225	26.2825	26.490	23.7329	24.083	20.1300	20.345	19.2257	19.319
	3	60.4151	60.453	50.8856	51.280	45.9425	46.605	38.9324	39.337	37.1805	37.355
	4	98.3411	98.402	82.8310	83.458	74.7714	75.822	63.2962	63.933	60.4428	60.718
	5	144.2895	144.38	121.5346	122.43	109.6874	111.18	92.7468	93.646	88.5575	88.946

Table 5. *Comparison of non-dimensional natural frequencies λ_i for CF beam*

		n=0.1		n=0.5		n=1		n=5		n=10	
L/h	Fq	Present	S&B[16]	Present	S&B[16]	Present	S&B[16]	Present	S&B[16]	Present	S&B[16]
5	1	1.7562	1.7574	1.4789	1.4911	1.3352	1.3557	1.1321	1.1446	1.0813	1.0867
	2	9.4924	9.5110	7.9815	8.0609	7.1906	7.3164	6.0562	6.1274	5.7856	5.8159
	3	14.0990	14.095	12.0400	12.012	10.8494	10.811	8.7814	8.7633	8.3503	8.3430
	4	22.6642	22.682	19.0756	19.243	17.1764	17.441	14.3710	14.516	13.7147	13.776
	5	37.7176	37.747	31.7521	32.022	28.5691	28.989	23.7865	24.009	22.6900	22.783
10	1	1.7954	1.7966	1.5120	1.5244	1.3655	1.3864	1.1594	1.1722	1.1074	1.1130
	2	10.7744	10.782	9.0719	9.1477	8.1884	8.3146	6.9351	7.0111	6.6234	6.6562
	3	28.1583	28.190	23.7474	24.024	21.3649	21.623	17.5049	17.527	16.6779	16.686
	4	28.4174	28.404	24.1860	24.098	21.8317	21.886	18.2274	18.391	17.3874	17.459
	5	51.5826	51.618	43.4392	43.787	39.1649	39.732	32.9360	33.2625	31.4357	31.575
20	1	1.8058	1.8070	1.5207	1.5332	1.3734	1.3945	1.1665	1.1795	1.1143	1.1199
	2	11.1886	11.196	9.4220	9.4992	8.5084	8.6383	7.2217	7.3014	6.8979	6.9324
	3	30.7797	30.800	25.9185	26.130	23.4003	23.755	19.8408	20.057	18.9501	19.394
	4	56.3733	56.379	47.9898	48.048	43.1764	43.246	35.0424	35.053	33.3679	33.372
	5	58.8655	58.897	49.6308	49.962	44.8203	45.402	37.8941	38.278	36.1786	36.345
30	1	1.8077	1.8089	1.5223	1.5348	1.3749	1.3960	1.1679	1.1809	1.1156	1.1212
	2	11.2710	11.278	9.4916	9.5691	8.5720	8.7027	7.2789	7.3594	6.9528	6.9876
	3	31.3047	31.325	26.3620	26.576	23.8056	24.165	20.2045	20.425	19.2986	19.394
	4	60.6407	60.681	51.0631	51.475	46.1043	46.795	39.1038	39.525	37.3497	37.532
	5	84.5695	84.569	72.0760	72.072	64.8758	64.870	52.5843	52.580	50.0599	50.058

4.4 The influence of the material distribution exponent n and the slenderness ratios L/h

Figures 4, 5, 6 show the changes of first three non-dimensional natural frequencies i of the FGM beam with SS, CC and CF boundary conditions with respect to different power law exponent n and ratio L/h. It is seen that all the frequencies decrease with the increase of n. Especially when n<1 all the frequencies decrease rapidly. Also, the natural frequencies increase when the ratio L/h increases for the fixed value of n. Change in the second and the third frequencies with respect to the ratio L/h is more pronounced than the change in the first frequency.

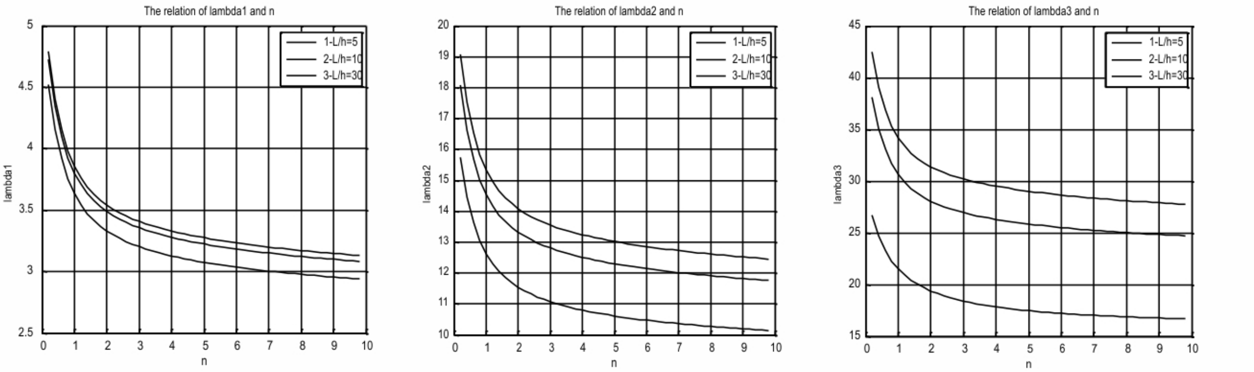

Figure 4. *Changes of first three non-dimensional natural frequencies of FGM beam for SS boundary condition on the different power law exponent n and ratios L/h*

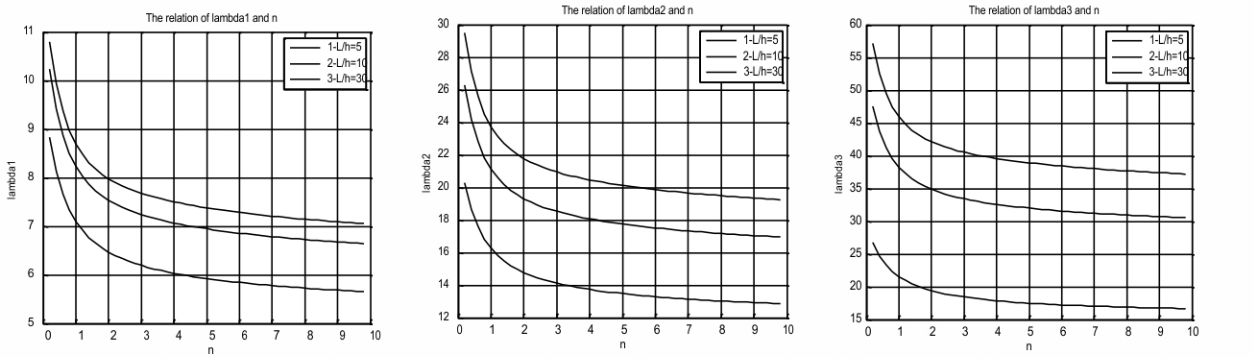

Figure 5. *Changes of first three non-dimensional natural frequencies of FGM beam for CC boundary condition on the different power law exponent n and ratios L/h*

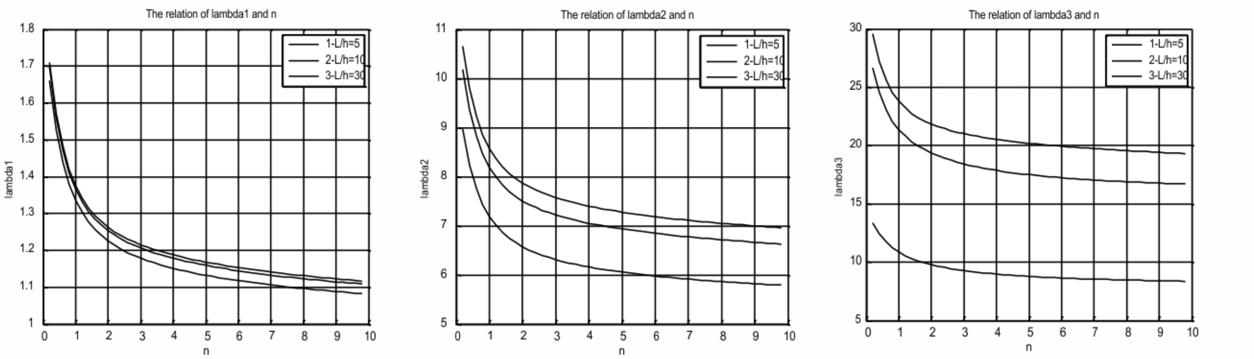

Figure 6. *Changes of first three non-dimensional natural frequencies of FGM beam for CF boundary condition on the different power law exponent n and ratios L/h*

5. Conclusions

In this paper, free vibration of FGM beam is investigated based on Timoshenko beam theory and power law distribution of FGM material using DSM. The DSM is recognized as the most accurate method in free vibration analysis, significantly superior to traditional FEM and other approximate methods. The method provides exact results for all natural frequencies and mode shapes without making any approximation on the way.

A consistent theory of vibration is established for FGM Timoshenko beam taking into account actual position of neutral axis that is a useful tool for analysis of coupled vibration in the beam. It is emphasize that when using neutral axis, the coefficients related to the mutual stiffness of axial and bending deformation are eliminated. Thence, the governing differential equations are in simpler form, which are similar to homogeneous beam. It allows one to propose an efficient method for free vibrations analysis of FGM Timoshenko beam from lowest natural frequencies.

The exactness of theoretical development has been illustrated and validated by numerical examples.

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