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RAYLEIGH WAVES IN COMPRESSIBLE ORTHOTROPIC HALF-SPACE OVERLAID BY A THIN UN-COAXIAL ORTHOTROPIC LAYER

Trinh Thi Thanh Hue[a,*], Phan Thi Thu Phuong[a], Pham Hong Anh[a]

[a] *Faculty of Faculty of Building and Industrial Construction, Hanoi University of Civil Engineering, 55 Giai Phong road, Hai Ba Trung district, Hanoi, Vietnam*

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Abstract

The problem of Rayleigh waves in compressible orthotropic elastic half-space overlaid by a thin elastic layer of which principal material axes are coincident have been researched by many scientists. However, the problem with the conditions that the half-space and the layer have only one common principal material axis that perpendicular to the layer while the remains are not identical has not gotten enough attention. This paper presents a traditional approach to obtain an approximate secular equation by approximately replacing the thin layer by effective boundary conditions of third-order. The wave then is considered as a Rayleigh wave propagating in an orthotropic half-space, without coating, subjected to the effective boundary conditions. This explicit approximate secular equation is potentially useful in non-damage assessment studies.

Keywords: orthotropic; monoclinic; principal material axis; thin layer; effective boundary condition; traditional approach.

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1. Introduction

Nowadays an elastic half-space overlaid by an elastic layer is a structure which has a lot of practical applications such as in acoustics, materials science, seismology, geophysics and microelectromechanical systems (MEMS). The measurement of mechanical properties of the layer deposited on the half-space before and during using is very important and necessary see [1] and references therein. Among various measurement methods, the surface wave method is used most widely [2], and for which the Rayleigh wave is a versatile and convenient tool.

By applying the Rayleigh wave tool, the explicit secular equation of Rayleigh waves are used as theoretical bases for taking out the mechanical properties of the thin layer for experimental data. Therefore, the main idea of the investigations of Rayleigh waves propagating in half-space coated by a thin layer is finding them. Taking the assumption of a thin layer, the explicit secular equation is derived by replacing approximately the full effect of the thin layer on the half-space by the so-called effective boundary conditions which relate the displacements with the stresses of the half-space at its surface.

To obtain the effective boundary conditions, we have ways either by replacing approximately the layer by a plate, see [3, 4], or by expanding the stresses at the upper surface of the layer into

[*]Corresponding author. *E-mail address:* huettt@nuce.edu.vn (Hue, T. T. T.)

Taylor series of the thickness of the layer, see [5–10]. Before the year 2010, there are some studies on Rayleigh waves in an elastic half-space coated by a thin layer such as Achenbach et al. [3], Bovik [5], Steigmann et al. [9], Tiersten [4], Tuan [11], and Wang et al. [12]. However, these studies are limited to isotropic materials. After the year 2010, Vinh et al. had many research into the structures of half-space overlaid by a thin layer with less symmetry substrates such as orthotropic [13–15] and monoclinic substrates [16, 17].

In all investigations mentioned above, the principal material axes of the half-space and of the thin layer were assumed to be coincide. Therefore, the main purpose of this paper is to present an explicit approximate secular equation of Rayleigh waves in a compressible orthotropic elastic half-space coated by a thin un-coaxial orthotropic layer. Specifically, the half-space and the thin layer have only one common principal material axis that perpendicular to the layer, the remains of the half-space are inclined at an angle θ to the remains of the layer. Here, the thin layer is assumed to be perfectly bonded to the half-space. From the relationship between principal material axes of the half-space and of the thin layer, we first transfer the original problem into the problem of Rayleigh waves propagation in a compressible orthotropic elastic half-space coated by a thin coaxial monoclinic layer with the symmetrical plane $x_3 = 0$. After that, the thin layer is approximately replaced by effective boundary conditions of third order. Then, the wave is considered as a Rayleigh wave propagating in the half-space, without the coated layer, subjected to the effective boundary conditions. By employing the traditional approach, the approximate secular equation is derived. This equation is then simplified for the case when the principal material axes of the half-space and of the layer were coincide. It is shown that this result coincides with the one previously obtained.

2. Formulation of problem

Consider an orthotropic half-space coated with a thin elastic orthotropic layer. Although the thin elastic layer is also orthotropic but its principal material axes do not coincide with those of the half-space. Suppose the principal material axes of the half-space are denoted by Ox_1, Ox_2, Ox_3 and the principal material axes of the layer are denoted by OX_1, OX_2, OX_3. However, the axes Ox_1, Ox_2 of the half-space are inclined at an angle θ to the axes OX_1, OX_2 of the layer and $Ox_3 \equiv OX_3$. We have

$$x_i = \Omega_{ij} X_j, \quad \Omega_{ij} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

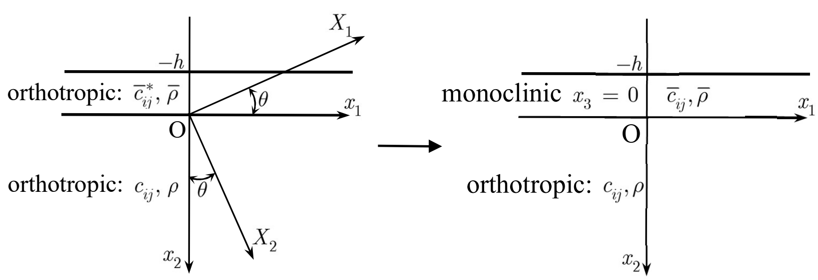

Figure 1. Model of Problem

In the layer, the material constants in the coordinates $OX_1X_2X_3$ are $\bar{c}^*_{ij}$ and the material constants in the coordinates $Ox_1x_2x_3$ are $\bar{c}_{ij}$. In the new coordinates $Ox_1x_2x_3$, the orthotropic layer becomes the layer made of a monoclinic material with the symmetrical plane $x_3 = 0$. However, in the paper, we consider the plane strain problem. Hence, we only use the following material constants are given by (see Ting [18])

$$\begin{aligned}
\bar{c}_{11} &= \bar{c}^*_{11}\cos^4\theta + 2\left(\bar{c}^*_{12} + 2\bar{c}^*_{66}\right)\cos^2\theta\sin^2\theta + \bar{c}^*_{22}\sin^4\theta \\
\bar{c}_{22} &= \bar{c}^*_{22}\cos^4\theta + 2\left(\bar{c}^*_{12} + 2\bar{c}^*_{66}\right)\cos^2\theta\sin^2\theta + \bar{c}^*_{11}\sin^4\theta \\
\bar{c}_{12} &= \bar{c}^*_{12} + \left(\bar{c}^*_{11} + \bar{c}^*_{22} - 2\bar{c}^*_{12} - 4\bar{c}^*_{66}\right)\cos^2\theta\sin^2\theta \\
\bar{c}_{16} &= -\left[\bar{c}^*_{11}\cos^2\theta - \bar{c}^*_{22}\sin^2\theta - \left(\bar{c}^*_{12} + 2\bar{c}^*_{66}\right)\left(\cos^2\theta - \sin^2\theta\right)\right]\cos\theta\sin\theta \\
\bar{c}_{26} &= \left[\bar{c}^*_{22}\cos^2\theta - \bar{c}^*_{11}\sin^2\theta - \left(\bar{c}^*_{12} + 2\bar{c}^*_{66}\right)\left(\cos^2\theta - \sin^2\theta\right)\right]\cos\theta\sin\theta \\
\bar{c}_{66} &= \bar{c}^*_{66} + \left(\bar{c}^*_{11} + \bar{c}^*_{22} - 2\bar{c}^*_{12} - 4\bar{c}^*_{66}\right)\cos^2\theta\sin^2\theta
\end{aligned} \tag{2}$$

So, the initial problem consider the propagation of a Rayleigh wave in an orthotropic elastic half-space coated by a thin un-coaxial orthotropic layer. Now, we consider the propagation of a Rayleigh wave in an orthotropic elastic half-space coated by a thin coaxial monoclinic elastic layer with the symmetrical plane $x_3 = 0$.

3. Effective boundary conditions

3.1. Basic equations of the elastic layer in matrix form

Consider an elastic half-space $x_2 \geq 0$ overlaid by a thin elastic layer $-h \leq x_2 \leq 0$. An elastic half-space is made of a orthotropic material and a thin elastic is made of a monoclinic material with the symmetrical plane $x_3 = 0$. For such materials, in-plane motions are decoupled from anti-plane motions, therefore we can consider the plane strain such that:

$$u_j = u_j(x_1, x_2, t), \quad \bar{u}_j = \bar{u}_j(x_1, x_2, t), \quad j = 1, 2, \quad u_3 \equiv \bar{u}_3 \equiv 0 \tag{3}$$

where u_j and $\bar{u}_j$ are the displacement components of the half-space and of the thin layer, t is the time.

Suppose that the layer is made of compressible monoclinic elastic material with the symmetrical plane $x_3 = 0$. Then the strain-stress relation are

$$\begin{aligned}
\bar{\sigma}_{11} &= \bar{c}_{11}\bar{u}_{1,1} + \bar{c}_{12}\bar{u}_{2,2} + \bar{c}_{16}\left(\bar{u}_{1,2} + \bar{u}_{2,1}\right) \\
\bar{\sigma}_{22} &= \bar{c}_{12}\bar{u}_{1,1} + \bar{c}_{22}\bar{u}_{2,2} + \bar{c}_{26}\left(\bar{u}_{1,2} + \bar{u}_{2,1}\right) \\
\bar{\sigma}_{12} &= \bar{c}_{16}\bar{u}_{1,1} + \bar{c}_{26}\bar{u}_{2,2} + \bar{c}_{66}\left(\bar{u}_{1,2} + \bar{u}_{2,1}\right)
\end{aligned} \tag{4}$$

where $\bar{\sigma}_{ij}$ are the stress components of the layer, $\bar{c}_{ij}$ are the material constants of the layer and are defined by (2). Here, commas denote differentiation with respect to spatial variables x_j.

Equations of motion without body forces are

$$\begin{aligned}
\bar{\sigma}_{11,1} + \bar{\sigma}_{12,2} &= \bar{\rho}\ddot{\bar{u}}_1 \\
\bar{\sigma}_{12,1} + \bar{\sigma}_{22,2} &= \bar{\rho}\ddot{\bar{u}}_2
\end{aligned} \tag{5}$$

with $\bar{\rho}$ is the mass density of the layer and a dot indicates differentiation with respect to t.

Solving Eqs. $(4)_2$ and $(4)_3$ for $\bar{u}_{1,2}$ and $\bar{u}_{2,2}$, we have

$$\begin{aligned}
\bar{u}_{1,2} &= \bar{n}_{66}\bar{\sigma}_{12} + \bar{n}_{26}\bar{\sigma}_{22} - \bar{r}_6\bar{u}_{1,1} - \bar{u}_{2,1} \\
\bar{u}_{2,2} &= \bar{n}_{26}\bar{\sigma}_{12} + \bar{n}_{22}\bar{\sigma}_{22} - \bar{r}_2\bar{u}_{1,1}
\end{aligned} \tag{6}$$

where

$$\bar{n}_{66} = \frac{\bar{c}_{22}}{\Delta},\ \bar{n}_{26} = -\frac{\bar{c}_{26}}{\Delta},\ \bar{n}_{22} = \frac{\bar{c}_{66}}{\Delta},\ \bar{r}_6 = -\frac{\bar{c}_{12}\bar{c}_{26} - \bar{c}_{22}\bar{c}_{16}}{\Delta}$$
$$\bar{r}_2 = \frac{\bar{c}_{12}\bar{c}_{66} - \bar{c}_{16}\bar{c}_{26}}{\Delta},\ \Delta = \bar{c}_{22}\bar{c}_{66} - \bar{c}_{26}^2 \tag{7}$$

Substituting $(4)_1$ into $(5)_1$ and taking into account (6) yields

$$\bar{\sigma}_{12,2} = \bar{\rho}\ddot{\bar{u}}_1 - \bar{\eta}\bar{u}_{1,11} - \bar{r}_6\bar{\sigma}_{12,1} - \bar{r}_2\bar{\sigma}_{22,1} \tag{8}$$

in which

$$\bar{\eta} = \bar{c}_{11} - \bar{r}_6\bar{c}_{16} - \bar{r}_2\bar{c}_{12} \tag{9}$$

From the second of Eq. (5) it follows

$$\bar{\sigma}_{22,2} = \bar{\rho}\ddot{\bar{u}}_2 - \bar{\sigma}_{12,1} \tag{10}$$

Eqs. (6), (8) and (10) take the matrix form as follows

$$\bar{\varsigma}' = \bar{\mathbf{M}}\bar{\varsigma} \tag{11}$$

where

$$\bar{\zeta} = \begin{bmatrix} \bar{U} \\ \bar{\Sigma} \end{bmatrix},\quad \bar{U} = \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix},\quad \bar{\Sigma} = \begin{bmatrix} \bar{\sigma}_{12} \\ \bar{\sigma}_{22} \end{bmatrix},\quad \bar{\mathbf{M}} = \begin{bmatrix} \bar{\mathbf{M}}_1 & \bar{\mathbf{M}}_2 \\ \bar{\mathbf{M}}_3 & \bar{\mathbf{M}}_4 \end{bmatrix} \tag{12}$$

in which the matrices $\bar{\mathbf{M}}_k$ are given by

$$\bar{\mathbf{M}}_1 = \begin{bmatrix} -\bar{r}_6\partial_1 & -\partial_1 \\ -\bar{r}_2\partial_1 & 0 \end{bmatrix},\quad \bar{\mathbf{M}}_2 = \begin{bmatrix} \bar{n}_{66} & \bar{n}_{26} \\ \bar{n}_{26} & \bar{n}_{22} \end{bmatrix}$$
$$\bar{\mathbf{M}}_3 = \begin{bmatrix} \bar{\rho}\partial_t^2 - \bar{\eta}\partial_1^2 & 0 \\ 0 & \bar{\rho}\partial_t^2 \end{bmatrix},\quad \bar{\mathbf{M}}_4 = \bar{\mathbf{M}}_1^T \tag{13}$$

with the symbol T signifies transpose of matrix, the prime means derivative with respect to x_2. Here, we use the notations $\partial_1 = \partial/\partial x_1, \partial_1^2 = \partial^2/\partial x_1^2, \partial_t^2 = \partial^2/\partial t^2$. Eq. (11) is called the matrix form of the plane strain for compressible monoclinic elastic solids with the symmetrical plane $x_3 = 0$.

From (11) and (12) it follows

$$\begin{bmatrix} \bar{U}^{(n)} \\ \bar{\Sigma}^{(n)} \end{bmatrix} = \bar{\mathbf{M}}^n \begin{bmatrix} \bar{U} \\ \bar{\Sigma} \end{bmatrix},\quad n = 1, 2, 3, ...,\ x_2 \in [-h, 0] \tag{14}$$

3.2. Effective boundary conditions of third-order

Since the layer is thin, h is small. Expanding $\bar{\Sigma}(-h)$ into Taylor series about $x_2 = 0$ up to the third-order of $-h$ leads to

$$\bar{\Sigma}(-h) = \bar{\Sigma}(0) - h\bar{\Sigma}'(0) + \frac{h^2}{2}\bar{\Sigma}''(0) - \frac{h^3}{6}\bar{\Sigma}'''(0) \tag{15}$$

Suppose that the surface $x_2 = -h$ is traction-free, we have $\bar{\Sigma}(-h) = 0$. Using Eq. (14) at $x_2 = 0$ into Eq. (15) yields

$$\left\{\mathbf{I} - h\bar{\mathbf{M}}_4 + \frac{h^2}{2}\left(\bar{\mathbf{M}}_3\bar{\mathbf{M}}_2 + \bar{\mathbf{M}}_4^2\right) - \frac{h^3}{6}\left[\left(\bar{\mathbf{M}}_3\bar{\mathbf{M}}_1 + \bar{\mathbf{M}}_4\bar{\mathbf{M}}_3\right)\bar{\mathbf{M}}_2 + \left(\bar{\mathbf{M}}_3\bar{\mathbf{M}}_2 + \bar{\mathbf{M}}_4^2\right)\bar{\mathbf{M}}_4\right]\right\}\bar{\Sigma}(0)$$
$$+\left\{-h\bar{\mathbf{M}}_3 + \frac{h^2}{2}\left(\bar{\mathbf{M}}_3\bar{\mathbf{M}}_1 + \bar{\mathbf{M}}_4\bar{\mathbf{M}}_3\right) - \frac{h^3}{6}\left[\left(\bar{\mathbf{M}}_3\bar{\mathbf{M}}_1 + \bar{\mathbf{M}}_4\bar{\mathbf{M}}_3\right)\bar{\mathbf{M}}_1 + \left(\bar{\mathbf{M}}_3\bar{\mathbf{M}}_2 + \bar{\mathbf{M}}_4^2\right)\bar{\mathbf{M}}_3\right]\right\}\bar{U}(0) = 0 \tag{16}$$

Since the thin layer is bonded perfectly to the half-space at the plane $x_2 = 0$, we derive $U(0) = \bar{U}(0)$ and $\Sigma(0) = \bar{\Sigma}(0)$. Thus, from (16) we have

$$\left\{\mathbf{I} - h\bar{\mathbf{M}}_4 + \frac{h^2}{2}\left(\bar{\mathbf{M}}_3\bar{\mathbf{M}}_2 + \bar{\mathbf{M}}_4^2\right) - \frac{h^3}{6}\left[\left(\bar{\mathbf{M}}_3\bar{\mathbf{M}}_1 + \bar{\mathbf{M}}_4\bar{\mathbf{M}}_3\right)\bar{\mathbf{M}}_2 + \left(\bar{\mathbf{M}}_3\bar{\mathbf{M}}_2 + \bar{\mathbf{M}}_4^2\right)\bar{\mathbf{M}}_4\right]\right\}\Sigma(0)$$
$$+\left\{-h\bar{\mathbf{M}}_3 + \frac{h^2}{2}\left(\bar{\mathbf{M}}_3\bar{\mathbf{M}}_1 + \bar{\mathbf{M}}_4\bar{\mathbf{M}}_3\right) - \frac{h^3}{6}\left[\left(\bar{\mathbf{M}}_3\bar{\mathbf{M}}_1 + \bar{\mathbf{M}}_4\bar{\mathbf{M}}_3\right)\bar{\mathbf{M}}_1 + \left(\bar{\mathbf{M}}_3\bar{\mathbf{M}}_2 + \bar{\mathbf{M}}_4^2\right)\bar{\mathbf{M}}_3\right]\right\}U(0) = 0 \tag{17}$$

Eq. (17) expresses the relation between the traction vector and the displacement vector of the half-space at the plane $x_2 = 0$. It is called the effective boundary condition of third-order in the matrix form. It replaces approximately the all influence of the thin layer on the half-space. Note that, when the principal material axes of the layer coincide with those of the half-space i.e. $\theta = 0$, $\bar{\mathbf{M}}_k$ $(k = 1, 2, 3, 4)$ in Eq. (17) are equivalent to M_k $(k = 1, 2, 3, 4)$ in Eq. (9) of Ref. [13].

Substituting (13) into (17) yields the effective boundary conditions of third-order in the component form, namely:

$$\begin{aligned}
&\sigma_{12} + h\left(\bar{r}_6\sigma_{12,1} + \bar{r}_2\sigma_{22,1} + \bar{\eta}u_{1,11} - \bar{\rho}\ddot{u}_1\right)\\
&+\frac{h^2}{2}\left[\bar{\rho}\bar{n}_{66}\ddot{\sigma}_{12} + \left(\bar{r}_6^2 + \bar{r}_2 - \bar{\eta}\bar{n}_{66}\right)\sigma_{12,11} + \bar{\rho}\bar{n}_{26}\ddot{\sigma}_{22} + \left(\bar{r}_2\bar{r}_6 - \bar{\eta}\bar{n}_{26}\right)\sigma_{22,11}\right.\\
&\left.-2\bar{\rho}\bar{r}_6\ddot{u}_{1,1} + 2\bar{\eta}\bar{r}_6u_{1,111} - \bar{\rho}\left(1 + \bar{r}_2\right)\ddot{u}_{2,1} + \bar{\eta}u_{2,111}\right] + \frac{h^3}{6}\left\{\bar{\rho}\left[\bar{n}_{26}\left(2 + \bar{r}_2\right) + 3\bar{n}_{66}\bar{r}_6\right]\ddot{\sigma}_{12,1}\right.\\
&+\left(\bar{r}_6^3 + 2\bar{r}_2\bar{r}_6 - 2\bar{\eta}\bar{n}_{26} - 3\bar{\eta}\bar{n}_{66}\bar{r}_6\right)\sigma_{12,111} + \bar{\rho}\left[\bar{n}_{22} + \left(\bar{n}_{22} + \bar{n}_{66}\right)\bar{r}_2 + 2\bar{n}_{26}\bar{r}_6\right]\ddot{\sigma}_{22,1}\\
&+\left(\bar{r}_2^2 + \bar{r}_2\bar{r}_6^2 - \bar{\eta}\bar{n}_{22} - \bar{\eta}\bar{n}_{66}\bar{r}_2 - 2\bar{\eta}\bar{n}_{26}\bar{r}_6\right)\sigma_{22,111} - \rho^2\bar{n}_{66}\ddot{u}_{1,tt} - \bar{\rho}\left[3\bar{r}_6^2 + \left(2 + \bar{r}_2\right)\bar{r}_2 - 2\bar{\eta}\bar{n}_{66}\right]\ddot{u}_{1,11}\\
&\left.+\bar{\eta}\left[3\bar{r}_6^2 + 2\bar{r}_2 - \bar{\eta}\bar{n}_{66}\right]u_{1,1111} - \rho^2\bar{n}_{26}\ddot{u}_{2,tt} - \bar{\rho}\left[\left(2 + \bar{r}_2\right)\bar{r}_6 - \bar{\eta}\bar{n}_{26}\right]\ddot{u}_{2,11} + 2\bar{\eta}\bar{r}_6u_{2,1111}\right\} = 0\\
&\text{at } x_2 = 0
\end{aligned} \tag{18}$$

$$\begin{aligned}
&\sigma_{22} + h\left(\sigma_{12,1} - \bar{\rho}\ddot{u}_2\right) + \frac{h^2}{2}\left[\bar{\rho}\bar{n}_{26}\ddot{\sigma}_{12} + \bar{r}_6\sigma_{12,11} + \bar{\rho}\bar{n}_{22}\ddot{\sigma}_{22} + \bar{r}_2\sigma_{22,11} - \bar{\rho}\left(1 + \bar{r}_2\right)\ddot{u}_{1,1} + \bar{\eta}u_{1,111}\right]\\
&+\frac{h^3}{6}\left\{\bar{\rho}\left(\bar{n}_{22} + \bar{n}_{66} + \bar{n}_{66}\bar{r}_2 + \bar{n}_{26}\bar{r}_6\right)\ddot{\sigma}_{12,1} + \left(\bar{r}_6^2 + \bar{r}_2 - \bar{\eta}\bar{n}_{66}\right)\sigma_{12,111} + \bar{\rho}\bar{n}_{26}\left(1 + 2\bar{r}_2\right)\ddot{\sigma}_{22,1}\right.\\
&+\left(\bar{r}_2\bar{r}_6 - \bar{\eta}\bar{n}_{26}\right)\sigma_{22,111} - \rho^2\bar{n}_{26}\ddot{u}_{1,tt} + \bar{\rho}\left[\bar{\eta}\bar{n}_{26} - \left(2 + \bar{r}_2\right)\bar{r}_6\right]\ddot{u}_{1,11} + 2\bar{\eta}\bar{r}_6u_{1,1111}\\
&\left.-\rho^2\bar{n}_{22}\ddot{u}_{2,tt} - \bar{\rho}\left(1 + 2\bar{r}_2\right)\ddot{u}_{2,11} + \bar{\eta}u_{2,1111}\right\} = 0\\
&\text{at } x_2 = 0
\end{aligned} \tag{19}$$

4. An approximate third-order secular equation of Rayleigh waves

For the compressible orthotropic half-space, the strain-stress relations are

$$\begin{aligned}\sigma_{11} &= c_{11}u_{1,1} + c_{12}u_{2,2}\\ \sigma_{22} &= c_{12}u_{1,1} + c_{22}u_{2,2}\\ \sigma_{12} &= c_{66}\left(u_{1,2} + u_{2,1}\right)\end{aligned} \tag{20}$$

with σ_{ij} and c_{ij} are respectively the stress components and the material constants of half-space. For the strain energy of the material to be positive definite, the material constants $c_{11}, c_{22}, c_{12}, c_{66}$ satisfy

$$c_{ii} > 0, \quad i = 1,2,6, \quad c_{11}c_{22} - c_{12}^2 > 0 \tag{21}$$

When the body forces are vanished, equations of motion have the following form

$$\begin{aligned}\sigma_{11,1} + \sigma_{12,2} &= \rho\ddot{u}_1\\ \sigma_{12,1} + \sigma_{22,2} &= \rho\ddot{u}_2\end{aligned} \tag{22}$$

where ρ is the mass density of the half-space.

Introducing (20) into (22) leads to

$$\begin{aligned}c_{11}u_{1,11} + c_{66}u_{1,22} + \left(c_{12} + c_{66}\right)u_{2,12} &= \rho\ddot{u}_1\\ c_{66}u_{2,11} + c_{22}u_{2,22} + \left(c_{12} + c_{66}\right)u_{1,12} &= \rho\ddot{u}_2\end{aligned} \tag{23}$$

At $x_2 = +\infty$ the decay conditions are

$$U(+\infty) = \Sigma(+\infty) = 0 \tag{24}$$

So, the unknown vectors $U = [u_1 \;\; u_2]^T, \Sigma = [\sigma_{12} \;\; \sigma_{22}]^T$ must satisfy Eq. (23), the effective boundary conditions (18), (19) and the decay condition (24).

Now we consider the propagation of Rayleigh wave, travelling with velocity $c\,(> 0)$ and wave number $k\,(> 0)$ in the x_1-direction and decaying in the x_2-direction. According to Vinh and Ogden [19] the displacement components of this Rayleigh wave which satisfy Eq. (23) and the decay condition (24) are given by

$$\begin{aligned}u_1 &= \left(B_1e^{-kb_1x_2} + B_2e^{-kb_2x_2}\right)e^{ik(x_1-ct)}\\ u_2 &= \left(\alpha_1B_1e^{-kb_1x_2} + \alpha_2B_2e^{-kb_2x_2}\right)e^{ik(x_1-ct)}\end{aligned} \tag{25}$$

where B_1, B_2 are constants to be determined from the effective boundary conditions (18) and (19), b_1, b_2 are roots of the bellow equation

$$c_{22}c_{66}b^4 + \left[\left(c_{12} + c_{66}\right)^2 + c_{22}\left(X - c_{11}\right) + c_{66}\left(X - c_{66}\right)\right]b^2 + \left(c_{11} - X\right)\left(c_{66} - X\right) = 0 \tag{26}$$

whose real parts are positive to ensure the decay condition, $X = \rho c^2$ and

$$\begin{aligned}\alpha_n &= i\beta_n, \; n = 1,2, \quad i = \sqrt{-1}\\ \beta_n &= \frac{b_n\left(c_{12} + c_{66}\right)}{c_{22}b_n^2 - c_{66} + X} = \frac{c_{11} - X - c_{66}b_n^2}{\left(c_{12} + c_{66}\right)b_n}\end{aligned} \tag{27}$$

From Eq. (26) we have

$$\begin{aligned}b_1^2 + b_2^2 &= -\frac{\left(c_{12} + c_{66}\right)^2 + c_{22}\left(X - c_{11}\right) + c_{66}\left(X - c_{66}\right)}{c_{22}c_{66}} := S\\ b_1^2b_2^2 &= \frac{\left(c_{11} - X\right)\left(c_{66} - X\right)}{c_{22}c_{66}} := P\end{aligned} \tag{28}$$

For a Rayleigh wave exists, b_1, b_2 have positive real parts. We deduce

$$0 < X < \min\{c_{11}, c_{66}\} \tag{29}$$

and

$$b_1 + b_2 = \sqrt{S + 2\sqrt{P}}, \quad b_1 b_2 = \sqrt{P} \tag{30}$$

Substituting (25) into $(20)_{2,3}$ yields

$$\begin{aligned} \sigma_{12} &= -kc_{66}\left\{(b_1+\beta_1) B_1 e^{-kb_1x_2} + (b_2+\beta_2) B_2 e^{-kb_2x_2}\right\} e^{ik(x_1-ct)} \\ \sigma_{22} &= ikc_{66}\left\{(c_{12}-c_{22}b_1\beta_1) B_1 e^{-kb_1x_2} + (c_{12}-c_{22}b_2\beta_2) B_2 e^{-kb_2x_2}\right\} e^{ik(x_1-ct)} \end{aligned} \tag{31}$$

Introducing Eqs. (25) and (31) into the effective boundary conditions (18) and (19) gives to two equations for B_1, B_2

$$\begin{aligned} f(b_1) B_1 + f(b_2) B_2 &= 0 \\ F(b_1) B_1 + F(b_2) B_2 &= 0 \end{aligned} \tag{32}$$

where

$$\begin{aligned} f(b_n) &= f_0(b_n) + f_1(b_n)\varepsilon + f_2(b_n)\frac{\varepsilon^2}{2} + f_3(b_n)\frac{\varepsilon^3}{6} \\ F(b_n) &= F_0(b_n) + F_1(b_n)\varepsilon + F_2(b_n)\frac{\varepsilon^2}{2} + F_3(b_n)\frac{\varepsilon^3}{6} \end{aligned} \tag{33}$$

with

$$f_0(b_n) = -c_{66}(b_n+\beta_n) \tag{34}$$

$$f_1(b_n) = \bar{X} - \bar{\eta} - \bar{r}_2(c_{12} - c_{22}b_n\beta_n) - i\bar{r}_6c_{66}(b_n+\beta_n) \tag{35}$$

$$\begin{aligned} f_2(b_n) = &\left[\bar{n}_{66}\left(\bar{X}-\bar{\eta}\right) + \bar{r}_2 + \bar{r}_6^2\right] c_{66}(b_n+\beta_n) - \left(\bar{X}-\bar{\eta}+\bar{r}_2\bar{X}\right)\beta_n \\ &- i\left[\bar{n}_{26}\left(\bar{X}-\bar{\eta}\right) + \bar{r}_2\bar{r}_6\right](c_{12}-c_{22}b_n\beta_n) + 2i\bar{r}_6\left(\bar{X}-\bar{\eta}\right) \end{aligned} \tag{36}$$

$$\begin{aligned} f_3(b_n) = &-\left(\bar{X}-\bar{\eta}\right)\left[3\bar{r}_6^2 + 2\bar{r}_2 + \bar{n}_{66}\left(\bar{X}-\bar{\eta}\right)\right] - \bar{r}_2^2\bar{X} \\ &+ \left[\left(\bar{X}-\bar{\eta}\right)(2\bar{r}_6\bar{n}_{26} + \bar{r}_2\bar{n}_{66} + \bar{n}_{22}) + \bar{r}_2\left(\bar{r}_6^2 + \bar{r}_2 + \bar{r}_2\bar{n}_{22}\bar{X}\right)\right](c_{12}-c_{22}b_n\beta_n) \\ &- i\left[\left(\bar{X}-\bar{\eta}\right)\left(\bar{n}_{26}\bar{X} + 2\bar{r}_6\right) + \bar{r}_2\bar{r}_6\bar{X}\right]\beta_n \\ &+ i\left[\bar{r}_6^3 + 2\bar{r}_2\bar{r}_6 + \bar{r}_2\bar{n}_{26}\bar{X} + \left(\bar{X}-\bar{\eta}\right)(3\bar{r}_6\bar{n}_{66} + 2\bar{n}_{26})\right] c_{66}(b_n+\beta_n) \end{aligned} \tag{37}$$

$$F_0(b_n) = c_{12} - c_{22}b_n\beta_n \tag{38}$$

$$F_1(b_n) = -\left[c_{66}(b_n+\beta_n) - \bar{X}\beta_n\right] \tag{39}$$

$$\begin{aligned} F_2(b_n) = &-\left(\bar{r}_2 + \bar{n}_{22}\bar{X}\right)(c_{12}-c_{22}b_n\beta_n) + \bar{X} - \bar{\eta} + \bar{r}_2\bar{X} \\ &- i\left(\bar{r}_6 + \bar{n}_{26}\bar{X}\right) c_{66}(b_n+\beta_n) \end{aligned} \tag{40}$$

$$\begin{aligned} F_3(b_n) = &\left[\bar{r}_6^2 + \bar{r}_2 + \bar{n}_{66}\left(\bar{X}-\bar{\eta}\right) + (\bar{r}_6\bar{n}_{26} + \bar{r}_2\bar{n}_{66} + \bar{n}_{22})\bar{X}\right] c_{66}(b_n+\beta_n) \\ &- \left[\bar{X}-\bar{\eta} + \left(2\bar{r}_2 + \bar{n}_{22}\bar{X}\right)\bar{X}\right]\beta_n + i\left[\left(\bar{X}-\bar{\eta}\right)\left(\bar{n}_{26}\bar{X} + 2\bar{r}_6\right) + \bar{r}_2\bar{r}_6\bar{X}\right] \\ &- i\left[\bar{n}_{26}\left(\bar{X}-\bar{\eta}\right) + \bar{r}_2\left(\bar{r}_6 + \bar{n}_{26}\bar{X}\right)\right](c_{12}-c_{22}b_n\beta_n) \end{aligned} \tag{41}$$

and $\varepsilon = kh$ is the dimensionless thickness of the layer, $\bar{X} = \bar{\rho}c^2$.

Due to $B_1^2 + B_2^2 \neq 0$, the determinant of system (32) must be vanish. This fact yields

$$f(b_1)F(b_2) - f(b_2)F(b_1) = 0 \tag{42}$$

Using (33)–(41) into (42) and taking into account (27), the approximate third-order secular equation in ε of Rayleigh waves is given as follows

$$A_0 + A_1\varepsilon + \frac{A_2}{2}\varepsilon^2 + \frac{A_3}{6}\varepsilon^3 + O\left(\varepsilon^4\right) = 0 \tag{43}$$

where

$$\begin{aligned} A_0 &= f_0(b_1)F_0(b_2) - f_0(b_2)F_0(b_1) \\ &= c_{66}\left[(c_{12} + c_{22}\beta_1\beta_2)(b_2 - b_1) + (c_{12} + c_{22}b_1b_2)(\beta_2 - \beta_1)\right] \end{aligned} \tag{44}$$

$$\begin{aligned} A_1 &= f_0(b_1)F_1(b_2) - f_0(b_2)F_1(b_1) + f_1(b_1)F_0(b_2) - f_1(b_2)F_0(b_1) \\ &= \bar{X}c_{66}(b_2\beta_1 - b_1\beta_2) - \left(\bar{X} - \bar{\eta}\right)c_{22}(b_2\beta_2 - b_1\beta_1) + i\bar{r}_6A_0 \end{aligned} \tag{45}$$

$$\begin{aligned} A_2 &= f_0(b_1)F_2(b_2) - f_0(b_2)F_2(b_1) + f_2(b_1)F_0(b_2) - f_2(b_2)F_0(b_1) \\ &\quad + 2\left(f_1(b_1)F_1(b_2) - f_1(b_2)F_1(b_1)\right) \\ &= -\left[\bar{n}_{22}\bar{X} + \bar{r}_6^2 + \bar{n}_{66}\left(\bar{X} - \bar{\eta}\right)\right]A_0 + 2\bar{X}\left(\bar{X} - \bar{\eta}\right)(\beta_2 - \beta_1) \\ &\quad + \left[\bar{r}_2\bar{X} - \left(\bar{X} - \bar{\eta}\right)\right]\left[(c_{66} - c_{12})(\beta_2 - \beta_1) + (c_{66} - c_{22}\beta_1\beta_2)(b_2 - b_1)\right] \\ &\quad + 2i\bar{r}_6\bar{X}c_{66}(b_2\beta_1 - b_1\beta_2) - 2i\bar{r}_6\left(\bar{X} - \bar{\eta}\right)c_{22}(b_2\beta_2 - b_1\beta_1) \end{aligned} \tag{46}$$

$$\begin{aligned} A_3 &= f_0(b_1)F_3(b_2) - f_0(b_2)F_3(b_1) + f_3(b_1)F_0(b_2) - f_3(b_2)F_0(b_1) \\ &\quad + 3\left(f_1(b_1)F_2(b_2) - f_1(b_2)F_2(b_1)\right) + 3\left(f_2(b_1)F_1(b_2) - f_2(b_2)F_1(b_1)\right) \\ &= -\left[\left(3\bar{n}_{66}\bar{X} - 2\right)\left(\bar{X} - \bar{\eta}\right) + \left(\bar{n}_{22}\bar{X} + 2\bar{r}_2 + 3\bar{r}_6^2\right)\bar{X}\right](b_2\beta_1 - b_1\beta_2) \\ &\quad + \left\{\left(\bar{X} - \bar{\eta}\right)\left[3\bar{n}_{22}\bar{X} + \bar{n}_{66}\left(\bar{X} - \bar{\eta}\right) + 2\bar{r}_2 + 3\bar{r}_6^2\right] - 2\bar{r}_2^2\bar{X}\right\}c_{22}(b_2\beta_2 - b_1\beta_1) \\ &\quad - i\left[3\bar{n}_{66}\bar{r}_6\left(\bar{X} - \bar{\eta}\right) + 3\bar{n}_{22}\bar{r}_6\bar{X} - \bar{n}_{26}\bar{r}_2\bar{X} + \bar{r}_6^3\right]A_0 + 6i\bar{r}_6\bar{X}\left(\bar{X} - \bar{\eta}\right)(\beta_2 - \beta_1) \\ &\quad - 2i\left[\left(\bar{X} - \bar{\eta}\right)\left(\bar{n}_{26}\bar{X} - \bar{r}_6\right) + \bar{r}_2\bar{r}_6\bar{X}\right]\left[c_{12}(\beta_2 - \beta_1) + c_{22}\beta_1\beta_2(b_2 - b_1)\right] \\ &\quad + 4i\left[2\bar{r}_2\bar{r}_6\bar{X} - \left(\bar{X} - \bar{\eta}\right)\left(\bar{n}_{26}\bar{X} + 2\bar{r}_6\right)\right]c_{66}(b_2 - b_1 + \beta_2 - \beta_1) \end{aligned} \tag{47}$$

By Eq. (27) we deduce

$$\begin{aligned} \beta_2 - \beta_1 &= -\frac{c_{11} - X + c_{66}b_1b_2}{(c_{12} + c_{66})b_1b_2}(b_2 - b_1), \quad \beta_1\beta_2 = \frac{c_{11} - X}{c_{22}b_1b_2} \\ b_2\beta_2 - b_1\beta_1 &= -\frac{c_{66}(b_1 + b_2)}{c_{12} + c_{66}}(b_2 - b_1) \\ \beta_2b_1 - \beta_1b_2 &= -\frac{(c_{11} - X)(b_1 + b_2)}{(c_{12} + c_{66})b_1b_2}(b_2 - b_1) \end{aligned} \tag{48}$$

Introducing (48) into Eqs. (44)–(47) yields

$$A_k = \gamma\bar{A}_k \ \ (k = 0, 1, 2, 3), \quad \gamma = \frac{c_{66}(b_2 - b_1)}{(c_{12} + c_{66})b_1b_2} \tag{49}$$

where

$$\bar{A}_0 = \left[\left(c_{12}^2 - c_{11}c_{22} + c_{22}X\right)b_1b_2 + X(c_{11} - X)\right] \tag{50}$$

$$\bar{A}_1 = \left[\bar{X}(c_{11} - X) + \left(\bar{X} - \bar{\eta}\right)c_{22}b_1b_2\right](b_1 + b_2) + i\bar{r}_6\bar{A}_0 \tag{51}$$

$$
\begin{aligned}
\bar{A}_2 = & -\left[\bar{n}_{22}\bar{X} + \bar{r}_6^2 + \bar{n}_{66}\left(\bar{X} - \bar{\eta}\right)\right]\bar{A}_0 \\
& + 2\left[c_{12}\left(\bar{r}_2\bar{X} - \bar{X} + \bar{\eta}\right) - \bar{X}\left(\bar{X} - \bar{\eta}\right)\right]b_1b_2 \\
& - 2\left[\bar{r}_2\bar{X} - \left(\bar{X} - \bar{\eta}\right) + \left(\bar{X} - \bar{\eta}\right)\frac{\bar{X}}{c_{66}}\right](c_{11} - X) \\
& + 2i\bar{r}_6\left[\bar{X}\left(c_{11} - X\right) + \left(\bar{X} - \bar{\eta}\right)c_{22}b_1b_2\right](b_1 + b_2)
\end{aligned}
\tag{52}
$$

$$
\begin{aligned}
\bar{A}_3 = & -\left\{\left[\left(\bar{X} - \bar{\eta}\right)\left(3\bar{n}_{22}\bar{X} + \bar{n}_{66}\bar{X} - \bar{n}_{66}\bar{\eta} + 2\bar{r}_2 + 3\bar{r}_6^2\right) - 2\bar{r}_2^2\bar{X}\right]c_{22}b_1b_2\right. \\
& \left. + \left[\left(3\bar{n}_{66}\bar{X} - 2\right)\left(\bar{X} - \bar{\eta}\right) + \left(\bar{n}_{22}\bar{X} + 2\bar{r}_2 + 3\bar{r}_6^2\right)\bar{X}\right](c_{11} - X)\right\}(b_1 + b_2) \\
& - i\left[3\bar{n}_{66}\bar{r}_6\left(\bar{X} - \bar{\eta}\right) + 3\bar{n}_{22}\bar{r}_6\bar{X} - \bar{n}_{26}\bar{r}_2\bar{X} + \bar{r}_6^3\right]\bar{A}_0 \\
& - i\left[2\left(\bar{X} - \bar{\eta}\right)\left(\bar{n}_{26}\bar{X} + 5\bar{r}_6\right) - 10\bar{r}_2\bar{r}_6\bar{X}\right]\left[c_{12}b_1b_2 - (c_{11} - X)\right] \\
& - 6i\bar{r}_6\bar{X}\left(\bar{X} - \bar{\eta}\right)\left(\frac{c_{11} - X}{c_{66}} + b_1b_2\right)
\end{aligned}
\tag{53}
$$

in which b_1b_2 and $b_1 + b_2$ are defined by (28) and (30). Removing the factor γ, Eq. (43) reduces

$$
\bar{A}_0 + \bar{A}_1\varepsilon + \frac{\bar{A}_2}{2}\varepsilon^2 + \frac{\bar{A}_3}{6}\varepsilon^3 + O\left(\varepsilon^4\right) = 0 \tag{54}
$$

This is the third-order approximate dispersion equation and it is entirely explicit.

When the principal material axes of the layer coincide the principal material axes of the half-space, i.e. $\theta = 0$. Then, $\bar{c}_{11} = \bar{c}^*_{11}$, $\bar{c}_{22} = \bar{c}^*_{22}$, $\bar{c}_{12} = \bar{c}^*_{12}$, $\bar{c}_{66} = \bar{c}^*_{66}$, $\bar{c}_{16} = \bar{c}_{26} = 0$. We have:

$$
\begin{aligned}
& \bar{r}_6 = \bar{n}_{26} = 0, \; \bar{n}_{22} = \frac{1}{\bar{c}^*_{22}}, \; \bar{n}_{66} = \frac{1}{\bar{c}^*_{66}} \\
& \bar{r}_2 = \frac{\bar{c}^*_{12}}{\bar{c}^*_{22}} := r_1, \; \bar{\eta} = \frac{\bar{c}^*_{11}\bar{c}^*_{22} - \left(\bar{c}^*_{12}\right)^2}{\bar{c}^*_{22}} := -r_3
\end{aligned}
\tag{55}
$$

$$
\begin{aligned}
& r_2 = r_1 + \frac{r_3}{\bar{c}^*_{66}}, \quad r_4 = r_1r_2 + \frac{r_3}{\bar{c}^*_{22}}, \quad r_5 = \frac{1 + r_1}{\bar{c}^*_{22}} + \frac{r_1}{\bar{c}^*_{66}} \\
& r_6 = (r_1 + r_2)\,r_3, \quad r_7 = r_1^2 + 2r_2, \quad r_8 = \frac{1 + r_1}{\bar{c}^*_{66}} + \frac{1}{\bar{c}^*_{22}}
\end{aligned}
\tag{56}
$$

Note that, $r_j\left(j = \overline{1,8}\right)$ in Eqs. (55) and (56) are equivalent to $r_j\left(j = \overline{1,8}\right)$ in Eq. (12) of Ref. [13]. Substituting (55) and (56) into (33) yields

$$
\begin{aligned}
f\left(b_n\right) = & - c_{66}\left(b_n + \beta_n\right) + \varepsilon\left[r_3 + \bar{X} - r_1\left(c_{12} - c_{22}b_n\beta_n\right)\right] \\
& + \frac{\varepsilon^2}{2}\left[c_{66}\left(b_n + \beta_n\right)\left(r_2 + \frac{\bar{X}}{\bar{c}^*_{66}}\right) - \beta_n\left(\bar{X} + r_3 + \bar{X}r_1\right)\right] \\
& + \frac{\varepsilon^3}{6}\left[\left(c_{12} - c_{22}b_n\beta_n\right)\left(r_4 + r_5\bar{X}\right) - r_6 - r_7\bar{X} - \frac{\bar{X}^2}{\bar{c}^*_{66}}\right]
\end{aligned}
\tag{57}
$$

$$
\begin{aligned}
F(b_n) = & \; c_{12} - c_{22} b_n \beta_n - \varepsilon \left[c_{66} (b_n + \beta_n) - \beta_n \bar{X} \right] \\
& + \frac{\varepsilon^2}{2} \left[\bar{X} + r_3 + \bar{X} r_1 - (c_{12} - c_{22} b_n \beta_n) \left(r_1 + \frac{\bar{X}}{\bar{c}^*_{22}} \right) \right] \\
& + \frac{\varepsilon^3}{6} \left[c_{66} (b_n + \beta_n) \left(r_2 + r_8 \bar{X} \right) - \beta_n \left(\bar{X} + r_3 + 2 r_1 \bar{X} + \frac{\bar{X}^2}{\bar{c}^*_{22}} \right) \right]
\end{aligned}
\tag{58}
$$

This result is equivalent the result which was derived by Vinh et al. [13] (see Eq. (22)). So, the approximate third-order secular equation of Rayleigh waves of this case coincide with the approximate secular equation of third-order which was obtained by Vinh et al. [13].

5. Conclusions

In this paper, the authors researched on the Rayleigh waves in the elastic half-space coated by the thin elastic layer. Here, both the half-space and the thin layer are assumed to be orthotropic and compressible. However, the principal material axes of the half-space do not coincide with those of the half-space. By transforming the coordinate system, we bring the original problem to the problem of Rayleigh waves propagation in the orthotropic elastic half-space coated by a thin coaxial monoclinic layer with the symmetrical plane $x_3 = 0$. After that, the effective boundary conditions of third order are given that replaces the full influence of the thin layer on the half-space. For using the effective boundary conditions, an approximate dispersion equation of third order of Rayleigh waves is derived in this paper. This is a completely new result. When the principal material axes of the half-space and of the layer were coincide, the secular equation in this paper reduces to those given by Vinh et al. [13]. Because the obtained secular equation is entirely explicit, it will be useful in practical applications.

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