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EFFECTS OF END-PLATE ON THE CRITICAL MOMENT OF I-SECTION CANTILEVER BEAM WITH FREE END RESTRAINED LATERALLY

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Abstract

Effects of end-plate on the lateral buckling of doubly symmetrical I-section cantilever beam with free end restrained laterally are analyzed using the software COMSOL and linear shell finite elements. The torsional stiffness of the end-plate prevents the free warping of flanges and decreases the warping effective length of the cantilever beam. A parametric study is conducted on 3231 cantilever beams under uniform bending to propose an approximative formula to determine the warping effective length factor which depends on the ratio between the torsional constant of the end-plate and the warping stiffness of the beam. The small standard deviation and high coefficient of determination show a very good correlation between analytical formulas and numerical results. Numerical applications are applied to analyze some cantilever beams subjected to uniform bending to demonstrate the reliability of the proposed formula and the effects of the end-plate on the enhancement of the global stability of cantilever beams with free end restrained laterally.

Keywords: cantilever lateral buckling; end-plate; edge stiffener; flexural-torsional buckling; warping effective length; warping effective length factor.

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1. Introduction

A cantilever is the beam with one end fixed and the other free. In practice, the free end of the cantilever is usually restrained laterally by a bracing system or another beam. Thus, the cantilever beam with free end restrained both for the lateral translation and the twisting is considered in this research. On the lateral buckling problem, this case can be seen as one end fixed and the other pinned for both lateral bending and twisting. That is, the lateral bending effective length and warping effective length factors are equal to 0.7 ($k = k_w = 0.7$). The pinned-end boundary condition for the twisting is satisfied when the torsional rotation is equal to zero and the end section is free to warp as shown in Fig. 1(a).

Considering a cantilever beam with end-plate tightly welded to the free end of the beam, when the flanges of the beam are warped, the end-plate is deformed and the torsional stiffness of the end-plate prevents partially the warping of the flanges as indicated in Fig. 1(b). Therefore, the boundary condition for the warping changes and the end section of the beam is no longer free to warp. Thus, the warping effective length factor is less than 0.7 ($k_w < 0.7$).

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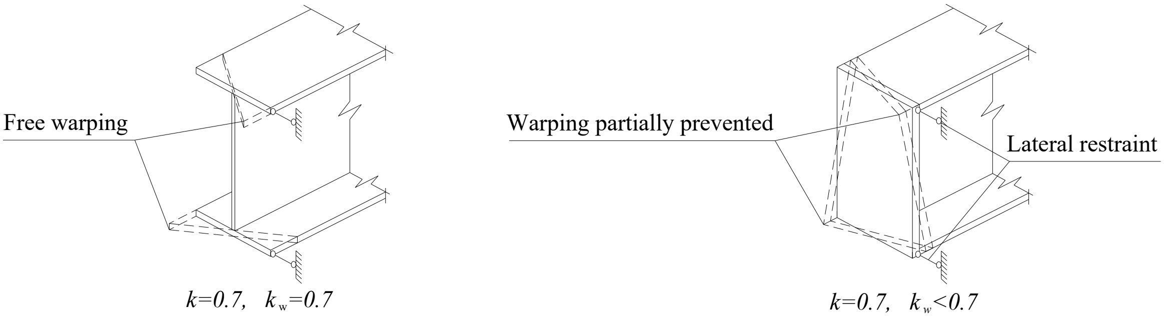

(a) Free end laterally restrained without end-plate

(b) Free end laterally restrained with end-plate

Figure 1. Warping of the end section of the cantilever

For simply supported beams, some studies analysed effects of the elastic restraint against the warping at the beam supports [1, 2]. Those studies theoretically demonstrate and experimentally verify the increase of the warping stiffness at the supports resulting in the increase of the torsional stiffness of the beam. Takabatake [3, 4] developed a lateral buckling theory for beams with web stiffeners and batten plates. In the theory, the web stiffeners and the batten plates are assumed to not have any impact on the warping. This assumption does not reflect the behavior of end-plates as shown in Fig. 1(b). Amara et al. [5] only paid attention to the torsion restraint but ignored the warping restraint of end connections. Piotrowski and Szychowski [6, 7] used the energy method including the energy of the elastic restraint against warping at the support sections to establish the expression of critical moment. The previous studies [1–7] are rather theoretical when there have not been any expressions for determining torsional stiffness at the end supports presented. Among others, Lindner and Gietzelt [8], Trahair [9], Pi and Trahair [10] proposed approximative expressions to calculate k_w which depends on the ratio of the torsional constant and the warping stiffness of the beam.

For cantilever beams, Hassanein et al. [11] proposed approximate formulas of exponential law to calculate coefficients representing the magnification of the critical moment due to using web stiffeners. However, they did not pay attention to the change of the warping effective length factor. Recently, Bui [12] established a formula to calculate the latter. However, the works of Hassanein et al. [11] and Bui [12] have not focused on the lateral buckling of cantilever beams with free end restrained laterally.

This research investigates effects of the torsional stiffness of the end-plate to the warping effective length factor of the cantilever beam with free end restrained laterally. The critical moment of the cantilever under uniform bending is calculated using both linear shell finite elements implemented in COMSOL and Djalaly's formula [13]. Firstly, the critical moment of the cantilever with end-plate is determined using COMSOL, then the warping effective length factor, k_w in Djalaly's formula is calculated from the numerical critical moment. A parametric study was conducted to find an approximative expression for the determination of k_w.

2. Critical moment formula

Some researchers proposed formulas to determine the critical moment of the beam. Among others, Djalaly [13] provided a general expression:

$$M_{cr} = C_1 \frac{\pi^2 EI_y}{(kL)^2} \left[\sqrt{\left(\frac{k}{k_w}\right)^2 \frac{I_w}{I_y} + \frac{(kL)^2 GI_t}{\pi^2 EI_y} + (C_2 y_g - C_3 y_j)^2} - (C_2 y_g - C_3 y_j) \right] \tag{1}$$

where L is the beam length; k and k_w are the bending effective length and warping effective length factors, respectively. C_1, C_2 and C_3 are coefficients which depend on load and boundary conditions. I_x, I_y, I_t and I_w are second moments of area about x and y axis, torsion constant, and warping constant, respectively. y_g is the distance between the loading point and the shear center:

$$y_g = y_a - y_s \tag{2}$$

where y_a is the ordinate of the loading point with-respect-to (w.r.t) the gravity center of the section; y_s is the ordinate of the shear center w.r.t the gravity center; y_j is calculated as:

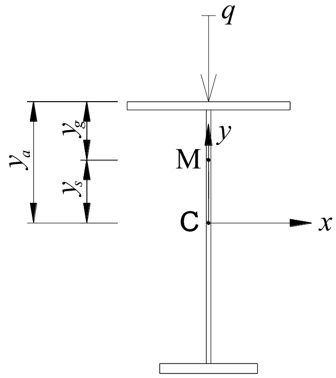

Figure 2. Mono-symmetrical I-section

$$y_j = y_s - \frac{1}{2I_x}\int_A y(x^2 + y^2)dA \tag{3}$$

The y ordinate and the geometrical properties are determined according to the section shown in Fig. 2.

Eq. (1) is one of the most commonly used formulas to estimate the critical moment. This equation was included in European pre-standard version (ENV) of the design standard Eurocode-1993 (EC3) [14]. The completed European Norm (EN) version of this design standard [15] does not provide information considering the determination of Mcr. In the ENV version [14], it is conservatively suggested to take the warping effective length factor, k_w equal to 1.0 unless special provisions for the warping restraint are provided.

For a beam of doubly symmetrical I-section: $y_s = 0$ and $y_j = 0$, hence Eq. (1) is reduced as following:

$$M_{cr} = C_1\frac{\pi^2 EI_y}{(kL)^2}\left[\sqrt{\left(\frac{k}{k_w}\right)^2\frac{I_w}{I_y} + \frac{(kL)^2 GI_t}{\pi^2 EI_y} + (C_2 y_g)^2} - C_2 y_g\right] \tag{4}$$

When loads are applied at the shear center of the doubly symmetrical I-section, i.e., $y_g = 0$, Eq. (4) is reduced to:

$$M_{cr} = C_1\frac{\pi^2 EI_y}{(kL)^2}\left(\sqrt{\left(\frac{k}{k_w}\right)^2\frac{I_w}{I_y} + \frac{(kL)^2 GI_t}{\pi^2 EI_y}}\right) \tag{5}$$

where C_1 is equal to 1.0 for the beam subjected to uniform bending.

3. Lateral buckling analysis on beams using linear shell finite elements and proposition of a formula to determine the warping effective length factor

For the simply supported I-section beam with end-plates, Lindner and Gietzelt [8], Trahair [9], Pi and Trahair [10] proposed approximate expressions for the calculation of k_w as below.

Lindner and Gietzelt's expression:

$$k_w = 1 - \frac{0.5}{1 + \frac{2}{\frac{\alpha_w L}{EI_w}}} \tag{6}$$

Trahair's expression:

$$k_w = \frac{4 + \frac{\alpha_w L}{EI_w}}{4 + 2\frac{\alpha_w L}{EI_w}} \tag{7}$$

Pi and Trahair's expression:

$$k_w = \frac{6.5 + \frac{\alpha_w L}{EI_w}}{6.5 + 2.25\frac{\alpha_w L}{EI_w}} \tag{8}$$

with

$$\alpha_w = \frac{1}{3}Gt_s^3 b_f d \tag{9}$$

α_w can be called torsional constant of the end plate; t_s, b_f, and d are respectively the thickness, width and height of the end-plate with a practical assumption that $b_f = b_s$ and $d = h_s$ (Fig. 3).

After a transformation, it can be recognized that Eqs. (6) and (7) are identical.

Based on the above expressions and the numerical simulation in COMSOL, a parametric study is performed. 3231 cantilever beams are simulated in COMSOL using linear shell elements; the end-moment is simulated using a couple of concentrated loads as shown in Fig. 3.

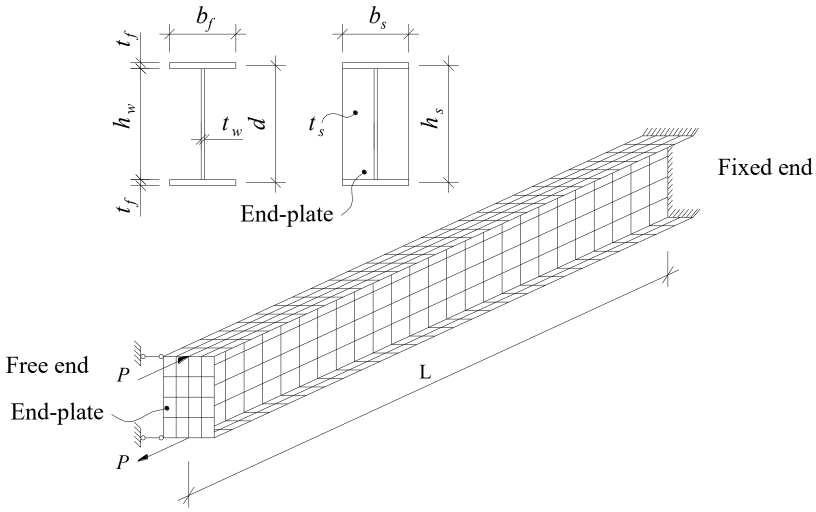

Figure 3. Simulation of the cantilever beam with free end restrained laterally

To validate the finite element (FE) model, a convergence study was conducted and is presented in Table 1. The convergence condition is satisfied when the mesh is finer.

Table 1. Validation of FE model

Section $d \times b_f \times t_f \times t_w$ (mm)	Span L (mm)	Number of elements for a flange	Number of elements for the web	Number of elements for the span	Number of elements for the end-plate	Total number of elements	M_{cr} (Tm)
$500 \times 250 \times 16 \times 10$	4000	2	3	30	6	216	318.39
		4	6	60	24	864	292.75
		6	9	90	54	1944	287.87
		8	12	120	96	3456	286.11
		10	15	150	200	5450	285.27

For the parametric study, 3231 cantilevers are selected with a range of dimensions as: $d = \{200, 300, 400, 500, 600, 700, 800, 900, 1000, 1200\}$, $b_f = \{200, 250, 300, 400, 500\}$, $t_f = \{14, 16, 18, 20, 22, 25, 28, 30\}$, $t_w = \{8, 10, 12, 14, 16\}$.

The thickness of the end-plate is taken: $t_s = \{8, 10, 12, 14, 16, 18, 20, 22, 25, 28, 30, 32, 36, 40, 45, 60\}$.

The selective cantilevers satisfy geometrical ratios as follows: $\dfrac{d}{b_f} = 1.0 \to 5.0$; $\dfrac{b_f}{t_f} = 8.0 \to 25.0$; $\dfrac{d}{t_w} = 20.0 \to 75.0$; $\dfrac{L}{d} = 4.0 \to 20.0$; $\dfrac{L}{b_f} = 15.0 \to 75.0$.

Firstly, the critical moment of the cantilever with end-plate is determined in COMSOL then the numerical warping effective length factor, $k_{w,num}$ can be determined using Eq. (10) after a transformation from Eq. (5) with

$$k_{w,num} = \frac{k}{\sqrt{\left[\left(\dfrac{M_{cr,COMSOL}(kL)^2}{\pi^2 EI_y}\right)^2 - \dfrac{(kL)^2 GI_t}{\pi^2 EI_y}\right]\dfrac{I_y}{I_w}}} \tag{10}$$

Finally, the formula for the determination of k_w is proposed as:

$$k_w = 0.7\frac{2.8 + n}{2.8 + 1.4n} \tag{11}$$

in which, the relative warping stiffness in this proposition is defined as:

$$n = \frac{\alpha_w (0.7L)}{EI_w} \tag{12}$$

where α_w is calculated from Eq. (9).

It is can be found that k_w calculated according to Eq. (11) yields a value ranging from 0.7 to 0.5 when the relative warping stiffness varies from zero to infinity.

Fig. 4 shows the correlation between the numerical results, $k_{w,num}$ analyzed using COMSOL (Eq. (10)) and the analytical curve plotted using Eq. (11).

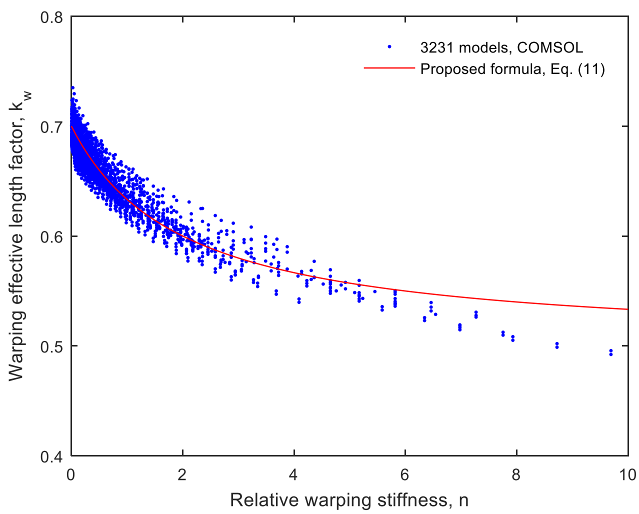

Figure 4. Correlation between numerical analysis and approximative formula

To evaluate the fit between the proposed formula (k_w) and numerical results ($k_{w,num}$), the standard deviation and coefficient of determination are calculated as follows:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{m}\left(k_{w,num,i} - \bar{k}_{w,i}\right)^2}{m-1}} = 0.0121 \tag{13}$$

$$R^2 = 1 - \frac{\sum_{i=1}^{m}\left(k_{w,num,i} - k_{w,i}\right)^2}{\sum_{i=1}^{m}\left(k_{w,num,i} - k_{w,i}\right)^2} = 0.9507 \tag{14}$$

where $k_{w,num,i}$ is the numerical value analyzed using COMSOL and Eq. (10); k_{wi} is the value calculated from the proposed formula (Eq. (11)) corresponding to the relative warping stiffness from Eq. (12); $\bar{k}_{w,i}$ is the mean value of $k_{w,num,i}$ and m is the total numerical models (m = 3231).

The value of the standard deviation, σ is 0.0121. This value is only about 2.0% when compared to the mean value of k_w in range of 0.7-0.5. The coefficient of determination, R^2 is 0.9507. Thus, there is a very good correlation between the proposed formula and numerical results.

Table 2. Critical moment of cantilever beams under uniform bending

Section $d \times b_f \times t_f \times t_w$ (mm)	Span L (mm)	Thick. of end-plate t_s, (mm)	n Eq. (12)	k_w Eq. (11)	M_{cr} Eq. (5) (Tm)	M_{cr} COMSOL (Tm)	Differ. (%)	M_{cr}/M_{ocr}
300×200×14×8	3000	-	0	0.70	142.37	138.20	3.02%	1.000
		8	0.0197	0.6981	142.71	140.77	1.38%	1.019
		16	0.1575	0.6854	144.97	143.15	1.27%	1.036
		22	0.4095	0.6660	148.60	146.09	1.72%	1.057
		30	1.0385	0.6316	155.63	151.69	2.60%	1.098
		40	2.4615	0.5897	165.41	160.51	3.05%	1.161
		60	8.3077	0.5388	179.41	177.53	1.06%	1.285
		80	19.69	0.5184	185.83	188.77	−1.56%	1.366
		100	38.46	0.5099	188.68	194.880	−3.18%	1.410
		120	66.4615	0.5058	190.06	194.882	−2.47%	1.410
600×200×20×14	4500	-	0	0.70	187.84	187.65	0.10%	1.000
		10	0.0202	0.6980	188.27	190.35	−1.09%	1.014
		20	0.1615	0.6851	191.11	193.15	−1.06%	1.029
		30	0.5452	0.6572	197.64	198.56	−0.46%	1.058
		40	1.2923	0.6215	206.93	206.40	0.26%	1.100
		60	4.3615	0.5629	224.99	225.26	−0.12%	1.200
		80	10.3385	0.5324	236.07	241.55	−2.27%	1.287
		100	20.19	0.5180	241.79	252.75	−4.34%	1.347
		120	34.8923	0.5108	244.77	259.93	−5.83%	1.385
1200×500×28×16	9000	-	0	0.70	1887.42	1873.42	0.75%	1.000
		12	0.0040	0.6996	1888.43	1888.98	−0.03%	1.008
		25	0.0361	0.6965	1896.43	1897.70	−0.07%	1.013
		36	0.1077	0.6898	1913.66	1912.35	0.07%	1.021
		45	0.2103	0.6810	1936.95	1930.57	0.33%	1.031
		60	0.4985	0.6601	1994.67	1975.34	0.98%	1.054
		100	2.3077	0.5929	2209.02	2165.15	2.03%	1.156
		130	5.07	0.5566	2346.72	2213.46	6.02%	1.182

* Note: M_{ocr} is the critical moment of the cantilever beam without end-plate analyzed using COMSOL.

4. Numerical applications

Table 2 shows the difference between the values of critical moments of cantilever beams under uniform bending, which are determined using Eq. (5) with k_w calculated from Eq. (11) and numerical analysis using COMSOL (Eq. (10)). The table also presents the ratio between critical moments of the beam with end-plate and the beam without end-plate (shown in the last column). It should be noted that these critical moments are all analyzed using COMSOL.

From Table 2, it can be seen that the difference between values of critical moments calculated using the proposed formula and COMSOL is small enough for the application in civil engineering. The comparison between critical moments of the cantilever beam with end-plate and the cantilever beam without end-plate in the last column of Table 2 shows that when the thickness of end-plate increases, the critical moment increases. The increase is significant, i.e. $M_{cr}/M_{ocr} > 1.05$ when the warping effective length factor, k_w is less than 0.66.

5. Conclusions

The end-plate prevents the warping of cantilever beam restrained laterally. The prevention leads to the reduction of the warping effective length factor and the increase of the critical moment. The research proposes an approximate formula for the determination of the warping effective length factor replacing effects of the end-plate. The small standard deviation and the high coefficient of determination show the very good correlation between the proposed formula and numerical results. The increase is significant when the effective warping length factor is less than 0.66.

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References

[1] Gotluru, B. P., Schafer, B. W., Peköz, T. (2000). Torsion in thin-walled cold-formed steel beams. *Thin-Walled Structures*, 37(2):127–145.

[2] Gosowski, B. (2007). Non-uniform torsion of stiffened open thin-walled members of steel structures. *Journal of Constructional Steel Research*, 63(6):849–865.

[3] Takabatake, H. (1988). Lateral buckling of I beams with web stiffeners and batten plates. *International Journal of Solids and Structures*, 24(10):1003–1019.

[4] Takabatake, H., Kusumoto, S., Inoue, T. (1991). Lateral buckling behavior of I beams stiffened with stiffeners. *Journal of Structural Engineering*, 117(11):3203–3215.

[5] Amara, S., Kerdal, D. E., Jaspart, J.-P. (2008). Effect of end connection restraints on the stability of steel beams in bending. *Advanced Steel Construction*, 243–259.

[6] Piotrowski, R., Szychowski, A. (2015). Lateral-torsional buckling of beams elastically restrained against warping at supports. *Archives of Civil Engineering*, 61(4):155–174.

[7] Piotrowski, R., Szychowski, A. (2019). Lateral Torsional Buckling of Steel Beams Elastically Restrained at the Support Nodes. *Applied Sciences*, 9(9):1944.

[8] Lindner, J., Gietzelt, R. (1984). Stabilisierung von Biegeträgern mit I-Profil durch angeschweißte Kopfplatten. *Der Stahlbau*, 53(3):69–74.

[9] Trahair, N. (1993). *Flexural-torsional buckling of structures*. CRC Press.

[10] Pi, Y.-L., Trahair, N. S. (2000). Distortion and warping at beam supports. *Journal of Structural Engineering*, 126(11):1279–1287.

[11] Hassanien, M., Bahaa, M., Sobhy, H., Hassan, A., Inoue, J. (2004). Effect of vertical web stiffeners on lateral torsional buckling behavior of cantilever steel I-beams. *Journal of Applied Mechanics*, 7:233–246.

[12] Cuong, B. H. (2019). Effects of end-plate on the critical moment of I-section cantilever beam. *Journal of Science and Technology in Civil Engineering (STCE) - NUCE*, 13(5V):20–27. (in Vietnamese).

[13] Djalaly, H. (1974). Calcul de la résistance ultime au déversement. *Construction Métallique*, 11(1):58–77.

[14] EC3 (1992). *Eurocode 3: Design of steel structures, Part 1-1: general rules and rules for buildings, ENV 1993-1-1*. European Committee Standardization, Brussels.

[15] EC3 (2005). *Eurocode 3: Design of steel structures, Part 1-1: general rules and rules for buildings, EN 1993-1-1*. European Committee Standardization, Brussels.