

LINEAR ANALYSIS OF A RECTANGULAR PILE UNDER VERTICAL LOAD IN LAYERED SOILS

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Abstract

In this paper, analytical and numerical solutions are developed for the pile with a rectangular cross-section under vertical load in layered soils. The rectangular cross-section is considered as a circular cross-section with a proposed formulation of equivalent radius. A number of bar elements models the pile and soil column below the pile tip while a series of independent springs distributed along the pile shaft with spring stiffness determined by properties of the corresponding soil layer models the surrounding soil. The method is based on energy principles and variational approach and the 1D finite element method is used in a pile displacement approximation. A new equation for modulus reduction appropriate for the rectangular pile is also developed to match the results of the proposed method to those of the three-dimensional (3D) finite element analyses. The proposed solution verified by comparing its results to the 3D finite element analyses and the comparisons are in excellent agreement.

Keywords: rectangular piles; variational; energy principle; vertical load; finite element.

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1. Introduction

Linear analysis of a single pile under vertical load is not appropriate in pile design but still useful in determining the equivalent stiffness of the pile-soil system for linear soil-foundation-structure interaction analysis (Chang and Nghiem [1]). Linear stiffness of the pile is also needed in developing the nonlinear relationship of load and settlement in a nonlinear analysis. In the literature, many researchers have developed analytical and numerical solutions for a vertically loaded pile. Poulos and Davis [2] analyzed the settlement behavior of a single axially loaded incompressible cylindrical pile in ideal elastic soil mass using Mindlin's equation. Butterfield and Banerjee [3] obtained the response of rigid and compressible single piles embedded in a homogeneous isotropic linear elastic medium by a rigorous analysis based on Mindlin's solutions for a point load in the interior of an ideal elastic medium. Banerjee and Davies [4] presented an approximate elastic analysis of single piles embedded in a soil of linearly increasing modulus with depth with the fundamental solution for point loads acting at the interface of a two-layer elastic half-space. Guo et al. [5] proposed an infinite layer model using a cylindrical coordinate system to solve the static problem of a pile under vertical load in an elastic half space. Lee et al. [6] investigated the behavior of axially loaded piles in layered soil in terms of effective stresses, using a rigorous elastic load transfer theory and following the technique of Muki and Sternberg. Ai et al. [7] extended the Sneddon and Muki solutions to solve elastostatic problems in

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multilayered elastic materials. Southcott and Small [8] used finite layer technique to idealize the soil as finite layers of infinite horizontal extent for the solutions of single pile and pile group. Randolph and Wroth [9] proposed a subgrade reaction formula in an approximate closed-form solution widely used in the practical field currently. Poulos [10] also presented a series of solutions and applied the analysis to three-layered soil. Guo and Randolph [11] based on the work of Randolph and Wroth [9] to consider the effects of non-homogeneity on the relationship between load transfer spring stiffness and elastic soil properties. Guo [12] extended the closed-form solutions by Guo and Randolph [11] to account for nonzero shear modulus at the ground surface. The solutions were expressed in modified Bessel functions of non-integer order. Lee and Small [13] proposed elastic method for axially loaded piles in finite-layered soil using a discrete layer analysis. The pile component is represented by one-dimensional two-noded elements. The response of the layered soil continuum component subjected to a system of interaction forces from pile elements acting on the soil at the pile-soil interface is computed by using the finite-layer method.

Rajapakse [14] presented an elastic solution based on a variational method coupled with an integral boundary representation. Vallabhan and Mustafa [15] proposed a simple closed-form solution for a drilled pier embedded in a two-layer elastic soil in which the pile tip sits on the surface of the first soil layer. The method based on energy principles with displacement field assumptions. Governing equations were obtained by minimizing a potential energy function and calculus of variations. Lee and Xiao [16], Seo et al. [17], Seo and Prezzi [18] and Salgado et al. [19] developed solution methods for a vertically axial loaded pile in multilayered soil based on a theory proposed by Rajapakse [14]. Basu et al. [20] and Seo et al. [17] applied the above theory to a pile with a rectangular cross-section.

In this paper, the author presents a simple solution in analyzing a single pile with a rectangular cross-section under vertical load in multilayered soil. The main differences between the previous solutions and the current solution are 1) equivalent pile radius; 2) new formulation for equivalent soil modulus.

2. Pile-soil model

A pile of length L_p and Young's modulus E_p with a rectangular cross-section of the dimensions $B_x \times B_y$ is shown in Figs. 1(a) and 1(b). Assumption can be made that the vertical displacements are equal at the same distance from the pile shaft in the radial direction. The perimeter of a displacement contour at a distance of Δr from the pile shaft is given by (Fig. 1(c)):

$$p = 2(B_x + B_y) + 2\pi\Delta r \quad (1)$$

The equivalent pile radius is defined as:

$$r_p = \frac{(B_x + B_y)}{\pi} \quad (2)$$

From Eq. (1), the following relationship can be made:

$$\Delta r = \frac{p}{2\pi} - \frac{(B_x + B_y)}{\pi} \quad (3)$$

The distance from the pile shaft in the radial direction is written in terms of equivalent pile radius, r_p and equivalent contour radius, r as:

$$\Delta r = r - r_p \quad (4)$$

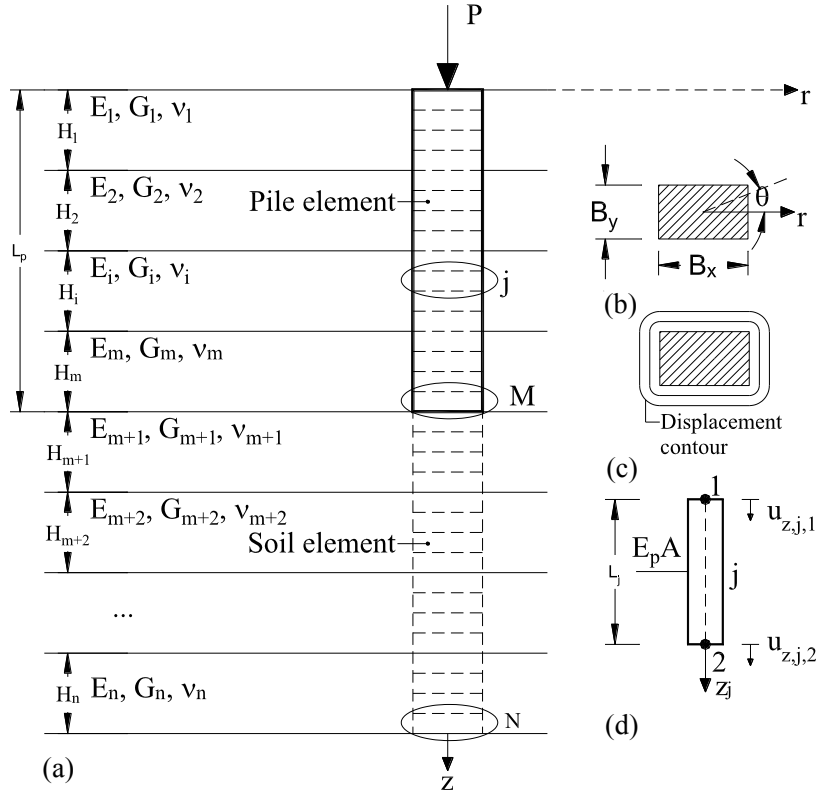


Figure 1. Pile-soil system

The pile is under axial load P applied at the center of the cross-section and embedded in a multi-layered soil medium with a total of n horizontal soil layers. The pile penetrates through m soil layers, and the pile base is assumed to be located at the bottom of the m^{th} layer then the pile base is underlain by $n - m$ soil layers. Properties of the i^{th} soil layer include Young's modulus, E_i , Poisson's ratio, ν_i , shear modulus, G_i and thickness, H_i . A bar element is used to model the pile and the soil column (below the pile tip), as shown in Fig. 1(a). The j^{th} pile element of length L_j (Fig. 1(d)) is inside the i^{th} soil layer and each soil layer surrounds several elements. A cylindrical coordinate system with its origin located at the center of the pile cross-section at the pile top with positive z -axis pointing downward coinciding with the pile axis. The pile and soil materials are assumed to be isotropic, homogeneous and linear elastic and the displacements at pile-soil interface compatible.

3. Displacement-strain-stress relationships

The assumption of the displacement field is proposed by Rajapakse [14]. Under vertical load, strains in the tangential direction are very small compared to the strains in the vertical direction and can be assumed negligible. The strain in the radial direction is also assumed negligible. Since the vertical displacement in radial direction decreases with increases in radial distance from the pile, the vertical displacement field in the soil can be approximated by a product of separable variables as:

$$u_z(r, z) = u_z(z) \phi(r) \quad (5)$$

where $u_z(z)$ is the vertical displacement of the pile at a depth of z ; $\phi(r)$ is the dimensionless function describing the reduction of the displacement in the radial direction from the pile center. It is assumed that $\phi(r) = 1$ at $r = r_p$ and $\phi(r) = 0$ at $r = \infty$.

With the above assumptions, the strain-displacement relationship is given by:

$$\begin{pmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{r\theta} \\ \gamma_{rz} \\ \gamma_{z\theta} \end{pmatrix} = \begin{pmatrix} -\frac{\partial u_r}{\partial r} \\ -\frac{u_r}{r} - \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \\ -\frac{\partial u_z}{\partial z} \\ -\frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \\ -\frac{\partial u_z}{\partial r} - \frac{\partial u_r}{\partial z} \\ -\frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\phi(r) \frac{du_z(z)}{dz} \\ 0 \\ -u_z(z) \frac{d\phi(r)}{dr} \\ 0 \end{pmatrix} \quad (6)$$

The relationships between stress and strain in the soil can be written in general form based on Hooke's law as follows:

$$\begin{pmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{r\theta} \\ \tau_{rz} \\ \tau_{z\theta} \end{pmatrix} = \begin{bmatrix} \lambda + 2G & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2G & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2G & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \begin{pmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{r\theta} \\ \gamma_{rz} \\ \gamma_{z\theta} \end{pmatrix} \quad (7)$$

where G and λ are Lamé's constants of the soil.

4. Governing equilibrium equations

The potential energy Π of the soil-pile system defined as the sum of internal energy and external energy can be expressed by:

$$\Pi = \sum_{j=1}^N \frac{1}{2} \int_0^{L_j} E_j A \left(\frac{du_{z,j}}{dz} \right)^2 dz + \sum_{j=1}^N \frac{1}{2} \int_0^{L_j} \int_0^{2\pi} \int_{r_p}^{\infty} \sigma_{kl} \varepsilon_{kl} r dr d\theta dz - P u_{z_0} \quad (8)$$

where E_j is Young's modulus of the j^{th} pile element, if $j \leq M$ then $E_j = E_p$, if $j > M$ then $E_j = E_i$; A is the area of the pile cross-section; $u_{z,j}$ is displacement of the j^{th} pile element; P and u_{z_0} are load and displacement at depth $z = z_0$, respectively.

Strain energy obtained by:

$$\frac{1}{2} \sigma_{kl} \varepsilon_{kl} = \frac{1}{2} (\lambda + 2G) \left(\phi \frac{du_z}{dz} \right)^2 + \frac{1}{2} G \left(u_z \frac{d\phi}{dr} \right)^2 \quad (9)$$

where σ_{kl} and ε_{kl} are the stress and the strain tensors.

By substituting Eq. (9) to Eq. (8), and integrating with respect to θ , potential energy can be obtained as:

$$\Pi = \sum_{j=1}^N \frac{1}{2} \int_0^{L_j} E_j A \left(\frac{du_{z,j}}{dz} \right)^2 dz + \sum_{j=1}^N \pi \int_0^{L_j} \int_{r_p}^{\infty} \bar{E}_i \left(\phi \frac{du_{z,j}}{dz} \right)^2 r dr dz + \sum_{j=1}^N \pi \int_0^{L_j} \int_{r_p}^{\infty} G_i \left(u_{z,j} \frac{d\phi}{dr} \right)^2 r dr dz - P u_{z0} \quad (10)$$

where $\bar{E}_i = \lambda_i + 2G_i$ is constraint modulus. Equilibrium equations of the soil-pile element can be made by minimizing the potential energy, or the first variation of the potential energy must be zero ($\delta\Pi = 0$).

The following differential equation for the pile element obtained by taking a variation on $u_{z,j}$:

$$\left(E_j A + 2\pi \int_{r_p}^{\infty} \bar{E}_i \phi^2 r dr \right) \frac{d^2 u_{z,j}}{dz^2} - \left[2\pi G_i \int_{r_p}^{\infty} \left(\frac{d\phi}{dr} \right)^2 r dr \right] u_{z,j} = 0 \quad (11)$$

Eq. (11) can be written in short form as:

$$(E_j A + t_j) \frac{d^2 u_{z,j}}{dz^2} - k_j u_{z,j} = 0 \quad (12)$$

where k_j and t_j are subgrade reactions for shearing and axial resistances, respectively, and determined by:

$$k_j = 2\pi G_i \int_{r_p}^{\infty} \left(\frac{d\phi}{dr} \right)^2 r dr \quad (13)$$

$$t_j = 2\pi \bar{E}_i \int_{r_p}^{\infty} \phi^2 r dr \quad (14)$$

5. Displacement approximation

According to the finite element method, vertical displacement in a bar element is approximated by nodal displacements as displayed in Fig. 1(c):

$$u_{z,j} = N_{j,1} u_{z,j,1} + N_{j,2} u_{z,j,2} \quad (15)$$

where $u_{z,j,1}$ and $u_{z,j,2}$ are vertical displacement at the first node and the second node of j^{th} pile element, respectively; $N_{j,1}$ and $N_{j,2}$ are shape functions. The shape functions can be obtained by using the following functions:

$$N_{j,1} = \frac{\cosh(\alpha_j z) \sinh(\alpha_j L_j)}{\sinh(\alpha_j L)} - \frac{\cosh(\alpha_j L_j) \sinh(\alpha_j z)}{\sinh(\alpha_j L)} \quad (16)$$

$$N_{j,2} = \frac{\sinh(\alpha_j z)}{\sinh(\alpha_j L_j)}$$

where z is the local coordinate of a pile element and α_j is calculated as:

$$\alpha_j = \sqrt{\frac{k_j}{E_j A + t_j}} \quad (17)$$

Applying the principle of minimum potential energy and taking a variation of ϕ , the governing differential equation for the soil surrounding the pile is given by:

$$\frac{d^2 \phi}{dr^2} + \frac{1}{r} \frac{d\phi}{dr} - \beta^2 \phi = 0 \quad (18)$$

where

$$\beta = \sqrt{\frac{b}{a}} \quad (19)$$

$$a = \sum_{j=1}^N G_i \int_0^{L_j} u_{z,j}^2 dz \quad (20)$$

$$b = \sum_{j=1}^N \bar{E}_i \int_0^{L_j} \left(\frac{du_{z,j}}{dz} \right)^2 dz \quad (21)$$

Based on the approximation of displacement in Eq. (15), values of a and b are calculated as:

$$a = \sum_{j=1}^N G_i \int_0^{L_j} (N_{j,1} u_{z,j,1} + N_{j,2} u_{z,j,2})^2 dz \quad (22)$$

$$= \sum_{j=1}^N \frac{G_i}{4\alpha_j \sinh^2(L_j \alpha_j)} \left\{ 4L_j \alpha_j u_{z,j,1} u_{z,j,2} \cosh(L_j \alpha_j) - 4u_{z,j,1} u_{z,j,2} \sinh(L_j \alpha_j) \right. \\ \left. - (u_{z,j,1}^2 + u_{z,j,2}^2) 2L_j \alpha_j + (u_{z,j,1}^2 + u_{z,j,2}^2) \sinh(2L_j \alpha_j) \right\}$$

$$b = \sum_{j=1}^N \bar{E}_i \int_0^{L_j} \left(\frac{dN_{j,1}}{dz} u_{z,j,1} + \frac{dN_{j,2}}{dz} u_{z,j,2} \right)^2 dz \quad (23)$$

$$= \sum_{j=1}^N \frac{\bar{E}_i \alpha_j}{4 \sinh^2(L_j \alpha_j)} \left\{ -4L_j \alpha_j u_{z,j,1} u_{z,j,2} \cosh(L_j \alpha_j) - 4u_{z,j,1} u_{z,j,2} \sinh(L_j \alpha_j) \right. \\ \left. + (u_{z,j,1}^2 + u_{z,j,2}^2) 2L_j \alpha_j + (u_{z,j,1}^2 + u_{z,j,2}^2) \sinh(2L_j \alpha_j) \right\}$$

The differential equation (18) is a form of the modified Bessel differential equation and its solution is given by:

$$\phi = c_1 I_0(\beta r) + c_2 K_0(\beta r) \quad (24)$$

where I_0 is a modified Bessel function of the first kind of zero-order, and K_0 is a modified Bessel function of the second kind of zero order. Apply the boundary conditions $\phi = 1$ at $r = r_p$, and $\phi = 0$ at $r = \infty$ to Eq. (24), solution of Eq. (18) leads to:

$$\phi = \frac{K_0(\beta r)}{K_0(\beta r_p)} \quad (25)$$

Subgrade reactions calculated by Eqs. (13) and (14) are written as follows:

$$k_j = 2\pi \int_{r_p}^{\infty} G_i \left(\frac{d\phi}{dr} \right)^2 r dr = \frac{\pi G_i r_p^2 \beta^2}{K_0^2(\beta r_p)} K_0(\beta r_p) K_2(\beta r_p) - \frac{\pi G_i r_p^2 \beta^2}{K_0^2(\beta r_p)} K_1^2(\beta r_p) \quad (26)$$

$$t_j = 2\pi \int_{r_p}^{\infty} \bar{E}_i \phi^2 r dr = \frac{\pi \bar{E}_i r_p^2}{K_0^2(\beta r_p)} [K_1^2(\beta r_p) - K_0^2(\beta r_p)] \quad (27)$$

An efficient solution of Eq. (11) based on the finite element method without solving a large number of equations proposed by Nghiem and Chang [21–23]. The solution provides displacements and axial forces along the pile.

6. Modification of soil moduli

The assumption that the displacement in the radial direction is equal to zero may result in pile response is stiffer than it is in reality. Near the pile head, the downdrag of the surrounding soil induces horizontal displacements toward the pile but this assumption restrains the displacement field in the horizontal direction. In fact, the term $\bar{E}_i = \lambda_i + 2G_i$ represents the soil constrained modulus, which is an indication that the analysis produces a stiff response. As the soil Poisson's ratio reaches to 0.5, the pile load-settlement response becomes increasingly stiffer while the constrained modulus is equal to infinity. Besides, stress only transfers from pile to soil in the radial direction also causes a stiff response. The 3D finite element model is more accurate because it covers all effects without any major assumption so it can consider the stress transfer to the soil in both vertical and radial directions. To eliminate the stiff response of the pile, Seo et al. [17] proposed a method by modifying the moduli of the soil by matching the pile responses obtained from their analyses with those obtained from finite element analyses. The moduli λ and G of the soil were replaced by λ^* and G^* , respectively, as the following equation (Seo et al. [17]):

For circular piles:

$$\lambda^* = 0 \quad \text{and} \quad G^* = 0.75G(1 + 1.25\nu^2) \quad (28)$$

For rectangular piles:

$$\lambda^* = 0 \quad \text{and} \quad G^* = 0.6G(1 + 1.25\nu^2) \quad (29)$$

Using above equations, the displacements along the pile did not match well with those from the 3D finite element analyses. In this study, the following equation adopted in the proposed solution which can produce the best matches to the 3D finite element analyses:

$$\lambda^* = 0 \quad \text{and} \quad G^* = 0.8G(1 + 1.25\nu^2) \quad (30)$$

7. Comparison with 3D finite element analyses

7.1. Pile in layered soil

The analyses have been performed using the proposed method in this study and the 3D finite element method using SSI3D program (Nghiem [24]). Two examples are considered and compared the analysis results with those the 3D finite element analyses. In the first example, parameters of the pile for the analyses are pile cross-section of $B_x \times B_y = 2.7 \text{ m} \times 1.2 \text{ m}$, pile length, $L_p = 30 \text{ m}$, and

pile modulus, $E_p = 25000$ MPa. A vertical load $P = 8000$ kN applied at the pile top. The pile is embedded in four-layer deposit with $H_1 = 2$ m, $H_2 = 10$ m, $H_3 = 10$ m (the pile base is in the fourth layer), $E_{s1} = 15$ MPa, $E_{s2} = 25$ MPa, $E_{s3} = 30$ MPa, $E_{s4} = 100$ MPa, $\nu_{s1} = 0.4$, $\nu_{s2} = 0.3$, $\nu_{s3} = 0.3$, $\nu_{s4} = 0.15$. In the finite element analyses, the soil-pile system is modeled by 15000 8-node solid elements. The lateral boundary is extended to 50 times of pile diameter, and the bottom boundary is extended to the depth equal to pile length below the pile tip. The axial displacements along the pile are shown in Fig. 3. Applying Eq. (30) to modify Young's modulus of the soil, the axial displacement curve is in excellent agreement with that of finite element analyses as shown in Fig. 2.

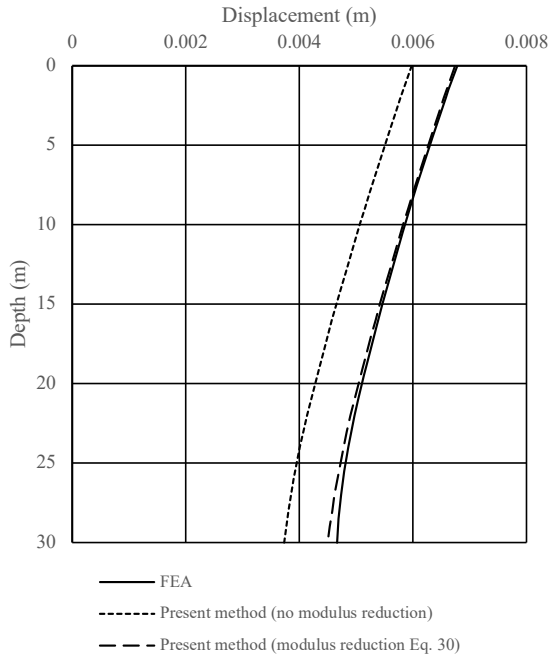


Figure 2. Displacement curves for example 1

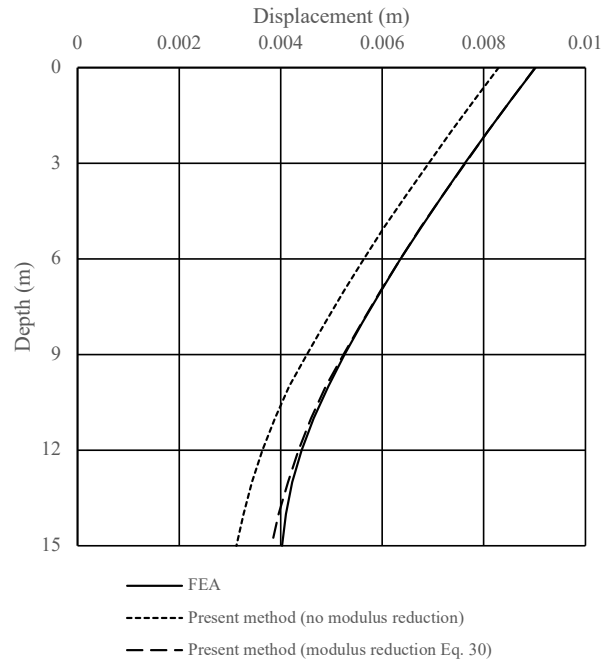


Figure 3. Displacement curves for example 2

7.2. Effects of aspect ratio

The aspect ratio effects on the behaviors of the rectangular piles were investigated by Seo et al. [17]. The aspect ratios of the cross-section of barrettes are usually greater than two (Fellenius et al. [25], Ng and Lei [26]). Seo et al. [17] showed that the effect of the aspect ratio on the normalized pile head stiffness was very small. In this study, the effect of the aspect ratio is also studied to verify the accuracy of the proposed solution. The pile in the first example is used in the analyses, $B_y = 1.2$ m is not changed, and B_x/B_y varies according to the following ratios: 1, 2.25, 3, and 4. The pile top displacement for each case of the analyses is plotted in Fig. 4. It can be seen that the aspect ratio effect on the accuracy of the proposed method is very small since the differences of the analysis results between the proposed method and FEA is not significant.

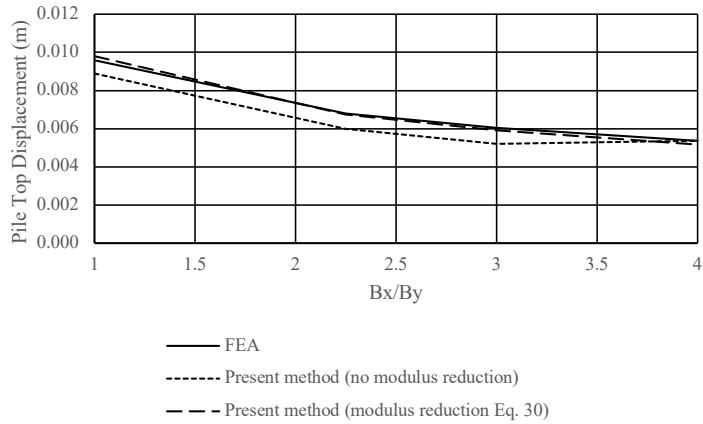


Figure 4. Effects of the aspect ratio on the pile response

7.3. Effects of Poisson's Ratio

The pile in the first example is also adopted in a parametric study to investigate the effects of Poisson's ratio on the pile response. Poisson's ratios of all soil layers are the same and vary from 0.1 to 0.49 in each analysis case. Fig. 5 shows the pile top and tip displacement versus Poisson's ratio. It

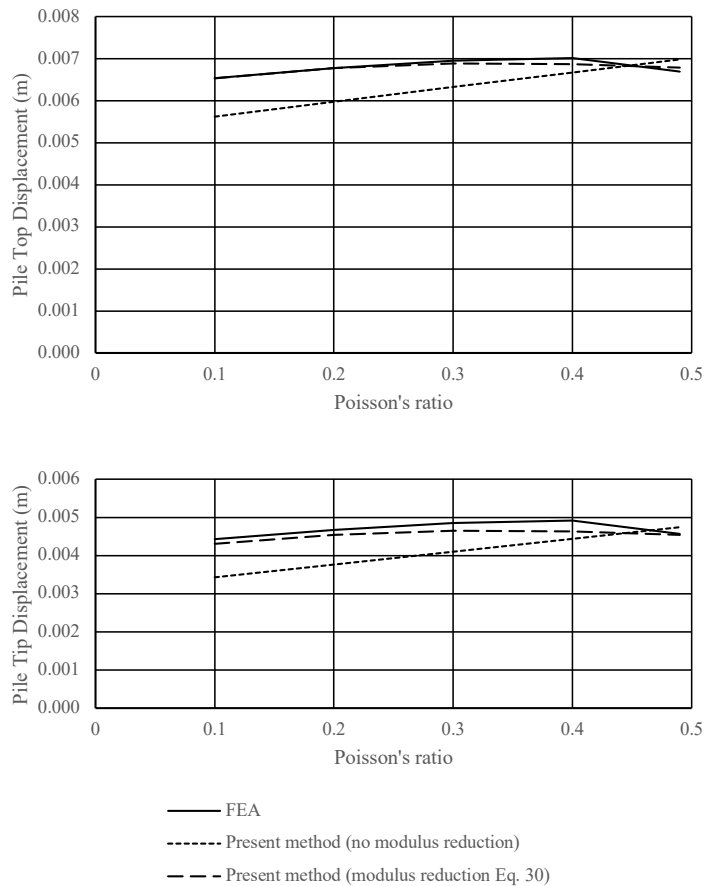


Figure 5. Effects of Poisson's ratio on the pile response

is evident that the modulus modification produces the pile top and tip displacements agrees very well to those of the finite element analyses (maximum differences are 2% at the pile top and 5.7% at the pile tip). The pile displacements for the analyses using original modulus values are closed to those of the finite element analyses only at Poisson's ratios greater than 0.4.

8. Conclusions

This paper presents a simple method for performance analysis of a single pile with a rectangular cross-section embedded in multiple layers of different soils under a vertical load at the pile head. The governing equations were derived based on continuum mechanics, strain energy and variational calculus by previous researchers. New formulations for the equivalent radius of the pile and modulus reduction for the rectangular piles are proposed. The analysis results using the new solution scheme compared well with the results from the 3D finite element analyses. The comparison of analysis results proves that using the new formulations are quite accurate for assessment of the pile performance for the rectangular piles embedded in multiple layers of different soils.

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