

# LARGE DISPLACEMENT ELASTIC ANALYSIS OF PLANAR STEEL FRAMES WITH FLEXIBLE BEAM-TO-COLUMN CONNECTIONS UNDER STATIC LOADS BY COROTATIONAL BEAM-COLUMN ELEMENT

Nguyen Van Hai<sup>a</sup>, Doan Ngoc Tinh Nghiem<sup>a</sup>, Le Van Binh<sup>a</sup>, Le Nguyen Cong Tin<sup>b</sup>,  
Ngo Huu Cuong<sup>a,\*</sup>

<sup>a</sup>*Faculty of Civil Engineering, Ho Chi Minh City University of Technology, Vietnam National University  
Ho Chi Minh City, 268 Ly Thuong Kiet street, District 10, Ho Chi Minh city, Vietnam*

<sup>b</sup>*Faculty of Civil Engineering, Mien Trung University of Civil Engineering,  
24 Nguyen Du street, Tuy Hoa city, Phu Yen province, Vietnam*

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## **Abstract**

This paper presents the elastic large-displacement analysis of planar steel frames with flexible connections under static loads. A corotational beam-column element is established to derive the element stiffness matrix considering the effects of axial force on bending moment (P- $\Delta$  effect), the additional axial strain caused by end rotations and the nonlinear moment – rotation relationship of beam-to-column connections. A structural nonlinear analysis program is developed by MATLAB programming language based on the modified spherical arc-length algorithm in combination with the sign of displacement internal product to automate the analysis process. The obtained numerical results are compared with those from previous studies to prove the effectiveness and reliability of the proposed element and program.

*Keywords:* corotational element; large-displacement analysis; flexible connections; steel frame; static loads; beam-column element.

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## **1. Introduction**

In practice, due to high slenderness of the steel members, the response of the steel structure is basically nonlinear. The effects of geometric nonlinearity and the flexibility of beam-to-column connections, which presents the nonlinear moment-rotation relationship of the connections, to the frame behavior are considerable, especially in large displacement analysis. There are three widespread formulations of element stiffness matrix of total Lagrangian, updated Lagrangian and co-rotational methods. In the co-rotational formulation, the local coordinate is attached to the element and simultaneously translates and rotates with the element during its deformation process. As a result, the derivation of the element stiffness matrix all relies on this local coordinate without the rigid body translation and rotation. Therefore, the co-rotational method reveals an outstanding advantage of dealing with large-displacement problems.

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\*Corresponding author. *E-mail address:* [ngohuucuong@hcmut.edu.vn](mailto:ngohuucuong@hcmut.edu.vn) (Cuong, N. H.)

Wempner [1], Belytschko and Glaum [2], Crisfield [3], Balling and Lyon [4], Le et al. [5], Nguyen [6], Doan-Ngoc et al. [7] and Nguyen-Van et al. [8] adopted the co-rotational method in their studies to predict the large-displacement behavior of the members and structures. However, the flexibility of the beam-to-column connections have not much paid attention in the combination with the co-rotational formulation. This study continues the work of Doan-Ngoc et al. for rigid steel frames with the consideration of the flexible connections. In this paper, a tangent hybrid element stiffness matrix is formed by performing partial derivative of force load vector with respect to local displacement variables. The flexible beam-to-column connections are modeled by zero-length rotational springs. The moment at flexible connections is updated during the analysis process upon the tangent rigidity and rotation. Notably, the proposed hybrid element is able to consider not only the P-delta effect but also the effect of axial strain caused by the bending of the element. The modified spherical arc-length which allows saving the computational effort on the basis that the stiffness matrix is only required to calculate for the first loop each load step is adopted. A sign criterion of product vector of displacement is combined with this non-linear equation solution method to trace the equilibrium path of structure. The obtained numerical results from the analysis program are compared to existing studies to illustrate the accuracy and efficiency of the proposed element.

**2. Finite element formulation**

*2.1. Internal force and rotation angle at element ends*

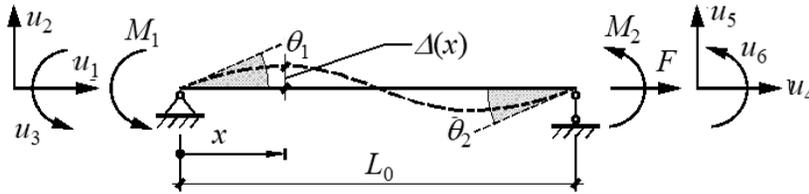


Figure 1. Co-rotational beam-column element

A traditional elastic beam-column element subjected to moment  $M_1$  and  $M_2$  at two extremities and axial force  $F$  is presented in Fig. 1. The displacement can be approximated via the function  $\Delta(x) = ax^3 + bx^2 + cx + d$  proposed by Balling and Lyon [4]. The relation of internal force and rotation at two ends can be expressed as:

$$\begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix} = \left( \frac{EI}{L_0} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} + FL_0 \begin{bmatrix} \frac{2}{15} & -\frac{1}{30} \\ -\frac{1}{30} & \frac{2}{15} \end{bmatrix} \right) \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} \tag{1}$$

$$F = \frac{EA}{L_0} \delta + EA \left[ \frac{1}{15} \theta_1^2 - \frac{1}{30} \theta_1 \theta_2 + \frac{1}{15} \theta_2^2 \right] \tag{2}$$

where  $\theta_1, \theta_2$  are rotational angle at two nodes of element.

*2.2. Internal force with consideration of connection flexibility*

Two zero-length springs are attached to two element nodes to form a hybrid beam-column element, as shown in Fig. 2. The rotation of the flexible connection will be:

$$\theta_1 = (\theta_{c1} - \theta_{r1}); \quad \theta_2 = (\theta_{c2} - \theta_{r2}) \tag{3}$$

where  $\theta_{ci}$  and  $\theta_i$  are the conjugate rotations for the moments  $M_{ci}$  and  $M_i$  at node  $i^{th}$ ;  $\theta_{ri}$  is incremental nodal rotations at node  $i^{th}$ .

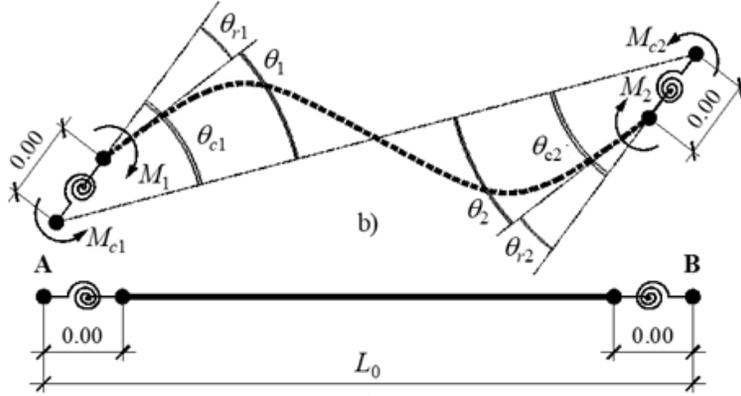


Figure 2. Beam-column element with flexible connection

The moment-rotation relation of flexible connection related to the tangent connection rigidities  $R_{kt1}$ ,  $R_{kt2}$  can be expressed in the incremental form:

$$\begin{cases} \Delta M_{c1} = R_{kt1} \Delta \theta_{r1} \\ \Delta M_{c2} = R_{kt2} \Delta \theta_{r2} \end{cases} \quad (4)$$

Meanwhile,

$$\begin{cases} M_{c1} = M_1 \\ M_{c2} = M_2 \end{cases} \quad (5)$$

Hence, the moment-rotation relation of flexible connection can be re-written as:

$$\begin{Bmatrix} \Delta M_{c1} \\ \Delta M_{c2} \end{Bmatrix} = \frac{EI}{L_0} \begin{bmatrix} s_{1c} & s_{2c} \\ s_{2c} & s_{3c} \end{bmatrix} \begin{Bmatrix} \Delta \theta_{c1} \\ \Delta \theta_{c2} \end{Bmatrix} \quad (6)$$

where  $s_{1c}$ ,  $s_{2c}$ ,  $s_{3c}$  are determined according to the tangent connection rigidities  $R_{kt1}$ ,  $R_{kt2}$ :

$$s_{1c} = \frac{\left(4 + 12 \frac{EI}{R_{kt2} L_0}\right)}{RR}, \quad s_{2c} = \frac{2}{RR}, \quad s_{3c} = \frac{\left(4 + 12 \frac{EI}{R_{kt1} L_0}\right)}{RR} \quad (7)$$

$$RR = \left(1 + \frac{4EI}{R_{kt1} L_0}\right) \left(1 + \frac{4EI}{R_{kt2} L_0}\right) - 4 \left(\frac{EI}{R_{kt1} L_0}\right) \left(\frac{EI}{R_{kt2} L_0}\right) \quad (8)$$

### 2.3. Co-rotational beam-column element stiffness matrix

The undeformed and deformed configuration of the co-rotational beam-column element AB is presented in Fig. 3. The local  $\bar{\mathbf{u}}$  displacement vector and the global displacement vector  $\mathbf{u}$  are:

$$\bar{\mathbf{u}} = \left\{ \delta \quad \theta_{c1} \quad \theta_{c2} \right\}^T, \quad \mathbf{u} = \left\{ u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \quad u_6 \right\}^T \quad (9)$$

The element length in two configurations  $L_0$  and  $L$ , respectively, is calculated as:

$$L_0 = \sqrt{(x_B - x_A)^2 + (z_B - z_A)^2}, \quad L = \sqrt{(x_B + u_4 - x_A - u_1)^2 + (z_B + u_5 - z_A - u_2)^2} \quad (10)$$

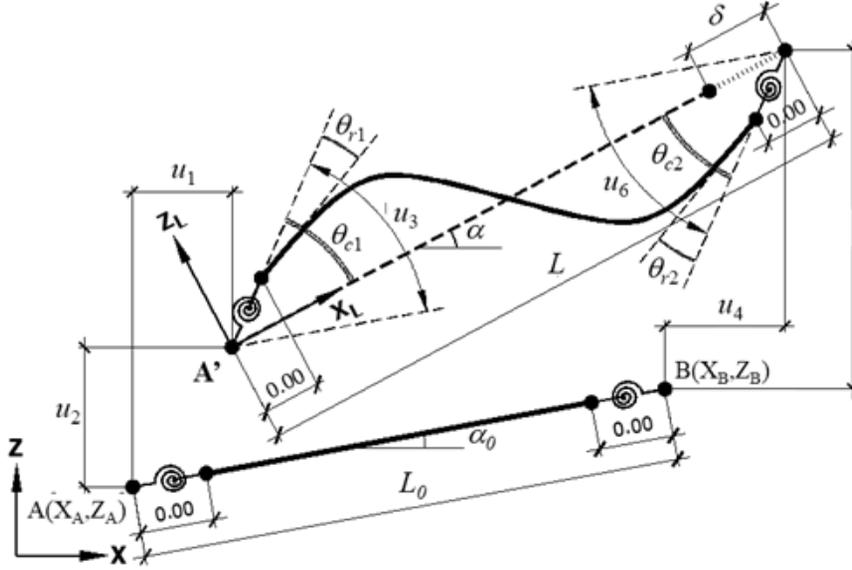


Figure 3. Initial and deformed configuration of beam-column element

The geometry parameter can be determined as:

$$\delta = (L - L_0), \quad \theta_{c1} = u_3 - (\alpha - \alpha_0), \quad \theta_{c2} = u_6 - (\alpha - \alpha_0) \quad (11)$$

$$\sin \alpha = \left( \frac{z_B + u_5 - z_A - u_2}{L} \right), \quad \cos \alpha = \left( \frac{x_B + u_4 - x_A - u_1}{L} \right) \quad (12)$$

$$\alpha_0 = \sin^{-1} \left( \frac{z_B - z_A}{L_0} \right), \quad \alpha = \sin^{-1} \left( \frac{z_B + u_5 - z_A - u_2}{L} \right) \quad (13)$$

Taking the derivative of  $\delta, \theta_{c1}, \theta_{c2}$  with respect to  $u_i$ , the global and local displacement relation is obtained as follows:

$$\left( \frac{\partial \bar{\mathbf{u}}}{\partial \mathbf{u}} \right) = \mathbf{B} = \begin{bmatrix} -\cos \alpha & -\sin \alpha & 0 & \cos \alpha & \sin \alpha & 0 \\ -\frac{\sin \alpha}{L} & \frac{\cos \alpha}{L} & 1 & \frac{\sin \alpha}{L} & -\frac{\cos \alpha}{L} & 0 \\ -\frac{\sin \alpha}{L} & \frac{\cos \alpha}{L} & 0 & \frac{\sin \alpha}{L} & -\frac{\cos \alpha}{L} & 1 \end{bmatrix} \quad (14)$$

Then, the relation of local element force  $\mathbf{f}_L$  and global element force  $\mathbf{f}_G$  is:

$$\mathbf{f}_L = \left\{ F \quad M_{c1} \quad M_{c2} \right\}^T \quad (15)$$

$$\mathbf{f}_G = \left\{ -F \quad \frac{(M_{c1} + M_{c2})}{L} \quad M_1 \quad F \quad -\frac{(M_{c1} + M_{c2})}{L} \quad M_2 \right\}^T \quad (16)$$

$$\mathbf{f}_G = \left( \frac{\partial \bar{\mathbf{u}}}{\partial \mathbf{u}} \right)^T \mathbf{f}_L = \mathbf{B}^T \mathbf{f}_L \quad (17)$$

Finally, the global tangent element stiffness matrix is achieved:

$$\mathbf{K}_G = \left( \frac{\partial \mathbf{f}_G}{\partial \mathbf{u}} \right) = \left( \frac{\partial \mathbf{B}^T}{\partial \mathbf{u}} \mathbf{f}_L + \mathbf{B}^T \frac{\partial \mathbf{f}_L}{\partial \mathbf{u}} \right) \quad (18)$$

$$\mathbf{K}_G = \mathbf{B}^T \mathbf{K}_L \mathbf{B} + \frac{\mathbf{r}_1 \mathbf{r}_1^T}{L} F + \frac{1}{L^2} \left[ \mathbf{r}_1 \mathbf{r}_2^T + \mathbf{r}_2 \mathbf{r}_1^T \right] (M_{c1} + M_{c2}) \quad (19)$$

where  $\mathbf{K}_L$  is local tangent element stiffness matrix

$$\mathbf{r}_1 = \left\{ \sin \alpha \quad -\cos \alpha \quad 0 \quad -\sin \alpha \quad \cos \alpha \quad 0 \right\}^T \quad (20)$$

$$\mathbf{r}_2 = \left\{ -\cos \alpha \quad -\sin \alpha \quad 0 \quad \cos \alpha \quad \sin \alpha \quad 0 \right\}^T \quad (21)$$

At connection positions,  $M_{c1} = M_1$ ,  $M_{c2} = M_2$ , thus the stiffness matrix  $\mathbf{K}_L$  is:

$$\mathbf{K}_L = \left( \frac{\partial \mathbf{f}_L}{\partial \bar{\mathbf{u}}} \right) = \begin{bmatrix} \frac{\partial F}{\partial \delta} & \frac{\partial M_{c1}}{\partial \delta} & \frac{\partial M_{c2}}{\partial \delta} \\ \frac{\partial F}{\partial \theta_{c1}} & \frac{\partial M_{c1}}{\partial \theta_{c1}} & \frac{\partial M_{c2}}{\partial \theta_{c1}} \\ \frac{\partial F}{\partial \theta_{c2}} & \frac{\partial M_{c1}}{\partial \theta_{c2}} & \frac{\partial M_{c2}}{\partial \theta_{c2}} \end{bmatrix} = \begin{bmatrix} \frac{\partial F}{\partial \delta} & \frac{\partial M_1}{\partial \delta} & \frac{\partial M_2}{\partial \delta} \\ \frac{\partial F}{\partial \theta_{c1}} & \frac{\partial M_1}{\partial \theta_{c1}} & \frac{\partial M_2}{\partial \theta_{c1}} \\ \frac{\partial F}{\partial \theta_{c2}} & \frac{\partial M_1}{\partial \theta_{c2}} & \frac{\partial M_2}{\partial \theta_{c2}} \end{bmatrix} \quad (22)$$

An explicit expression of  $\mathbf{K}_L$ :

$$K_{L(1,1)} = \frac{\partial F}{\partial \delta} = \frac{EA}{L_0} \quad (23)$$

$$K_{L(1,2)} = \frac{\partial M_{c1}}{\partial \delta} = \frac{\partial M_1}{\partial \delta} = EAH_1 \quad (24)$$

$$K_{L(1,3)} = \frac{\partial M_{c2}}{\partial \delta} = \frac{\partial M_2}{\partial \delta} = EAH_2 \quad (25)$$

$$K_{L(2,2)} = \frac{\partial M_{c1}}{\partial \theta_{c1}} = \frac{\partial M_1}{\partial \theta_{c1}} = \left( 4 \frac{EI}{L_0} + EAL_0 H_1^2 + \frac{2}{15} FL_0 \right) \quad (26)$$

$$K_{L(2,3)} = \frac{\partial M_{c2}}{\partial \theta_{c1}} = \frac{\partial M_2}{\partial \theta_{c1}} = \left( 2 \frac{EI}{L_0} + EAL_0 H_1 H_2 - \frac{1}{30} FL_0 \right) \quad (27)$$

$$K_{L(3,3)} = \frac{\partial M_{c2}}{\partial \theta_{c2}} = \frac{\partial M_2}{\partial \theta_{c2}} = \left( 4 \frac{EI}{L_0} + EAL_0 H_2^2 + \frac{2}{15} FL_0 \right) \quad (28)$$

$$K_{L(i,j)} = K_{L(j,i)} \quad (29)$$

where

$$H_1 = \left[ \frac{2}{15} (\theta_{c1} - \theta_{r1}) - \frac{1}{30} (\theta_{c2} - \theta_{r2}) \right] \quad (30)$$

$$H_2 = \left[ -\frac{1}{30} (\theta_{c1} - \theta_{r1}) + \frac{2}{15} (\theta_{c2} - \theta_{r2}) \right] \quad (31)$$

#### 2.4. Algorithm of nonlinear equation solution

The residual load vector at the loop  $i^{th}$  of the  $j^{th}$  load step is defined as

$$\mathbf{R}_j^{i-1} = \mathbf{F}_{in}^{i-1} - \lambda_j^{i-1} \mathbf{F}_{ex} \quad (32)$$

where  $\mathbf{F}_{in}$  is the system internal force vector which is accumulated global element force vector  $\mathbf{f}$ ,  $\mathbf{F}_{ex}$  is called the reference load vector and  $\lambda$  is load parameter. In order to solve the equation (32) continuously at “snap-back” and “snap-through” behavior, the modified arc-length nonlinear algorithm in

combination with the scalar product criterion, proposed by Posada [9], is adopted. Specifically, the sign of incremental load parameter  $\Delta\lambda_j^1$  at the first iteration of each incremental load level is

$$\Delta\lambda_j^1 = \pm \frac{\Delta s_j}{\sqrt{(\delta\hat{\mathbf{u}}_j^1)^T (\delta\hat{\mathbf{u}}_j^1)}} \quad (33)$$

$$\text{sign}(\Delta\lambda_j^1) = \text{sign}\left(\left(\{\Delta\mathbf{u}\}_{j-1}^{\text{satisfied}}\right)^T \{\delta\hat{\mathbf{u}}\}_j^1\right) \quad (34)$$

where  $\Delta\lambda_j^1$  and  $\{\Delta\mathbf{u}\}_{j-1}^{\text{satisfied}}$  are the incremental load factor at the  $j^{\text{th}}$  load step and the converged incremental displacement vector at the previous load step,  $\delta\hat{\mathbf{u}}_j^1 = \mathbf{K}_j \mathbf{F}_{ex}$  is the current tangential displacement vector.

### 3. Numerical examples

An automatic structural analysis MATLAB program is developed to trace the load-displacement behavior of steel frames with rigid or flexible connections under static loads. The efficiency of the coded program is verified through the comparison between the achieved results and those from preceding investigations in the three following examples.

#### 3.1. Linear flexible base column subjected to eccentric load

Fig. 4 presents a column with the applied loads, geometrical and material properties. The base is considered as a clamped point or a flexible connection with the rigidity of  $R_k$ . This member was investigated by So and Chan [10] by using two three-node elements with a four-order approximate function for the horizontal displacement. It can be seen in Fig. 5 that two proposed elements are adequate to achieve a good convergence for both column-base connection cases. The analytical results have a very good agreement with those of So and Chan (Fig. 6). Furthermore, this example illustrates the capacity of the developed program for dealing with the “snap-back” behavior.

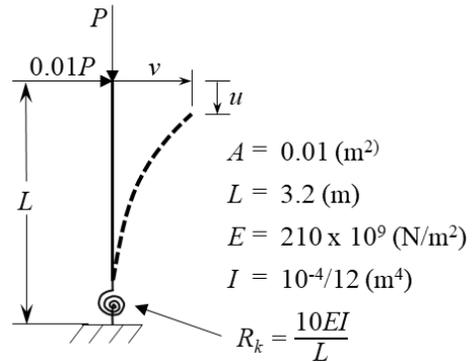


Figure 4. Column under eccentric load

#### 3.2. Cantilever beam with concentrated load at free end

A flexible base cantilever beam with a point load at the free end (Fig. 7) was studied by Aristizábal-Ochoa [11] using classical elastic method. The behavior of the moment-rotation relation of flexible connection is stimulated by the three-parameter model with ultimate moment  $M_u = EI/L$ , initial rotational angle  $\varphi_0 = 1$  and the factor  $n = 2$ . As shown in Fig. 8, the convergent load-displacement can be found with two proposed elements. The results from the written analysis program match very well with the analytical solution of Aristizábal-Ochoa (Fig. 9). In addition, it can be referred that the effect of connection flexibility is considerable. Specifically, at the load factor of 2, the non-dimensional displacement  $(1 - v/L)$  of the rigid beam is roughly 0.41 which much lower than that, 0.82, for the beam with flexible base.

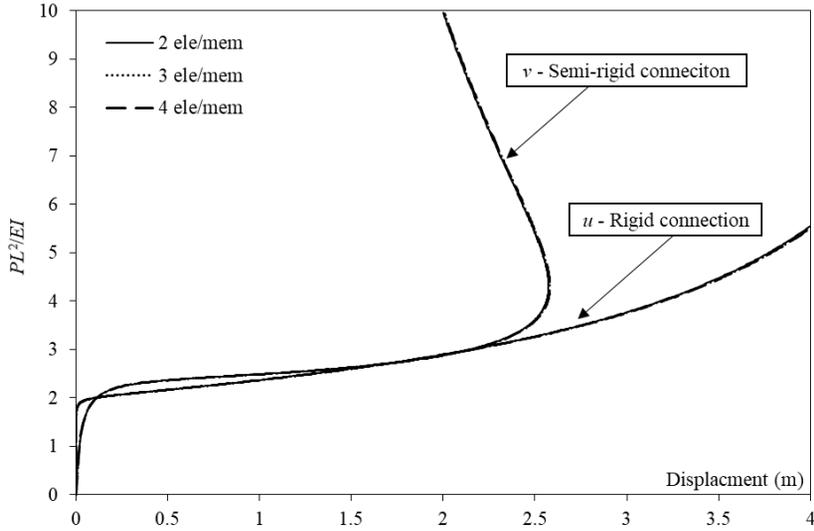


Figure 5. Convergence rate according to different number of proposed elements

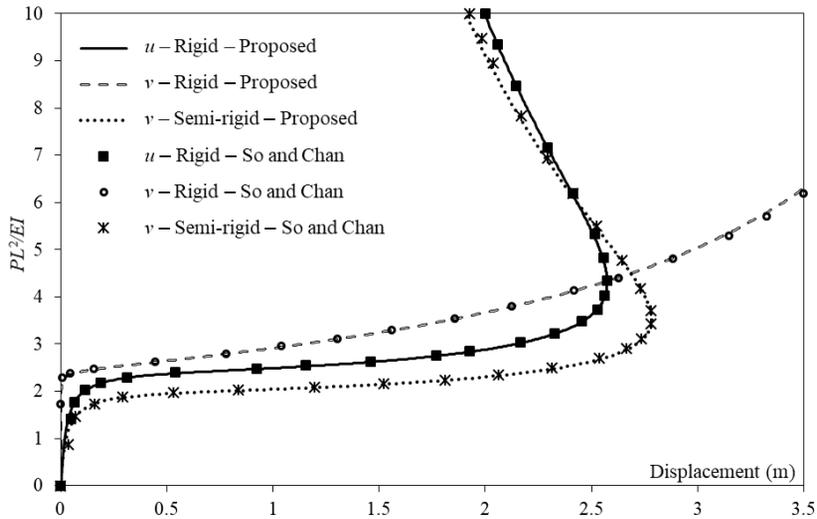


Figure 6. Load-displacement at column top

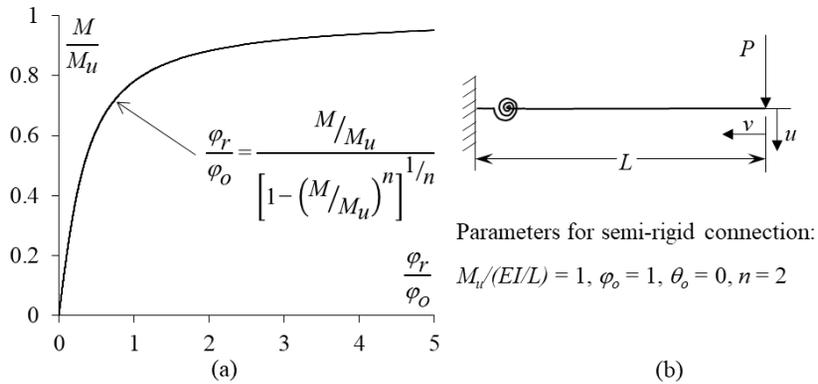


Figure 7. (a) moment-rotational relation model (b) cantilever beam

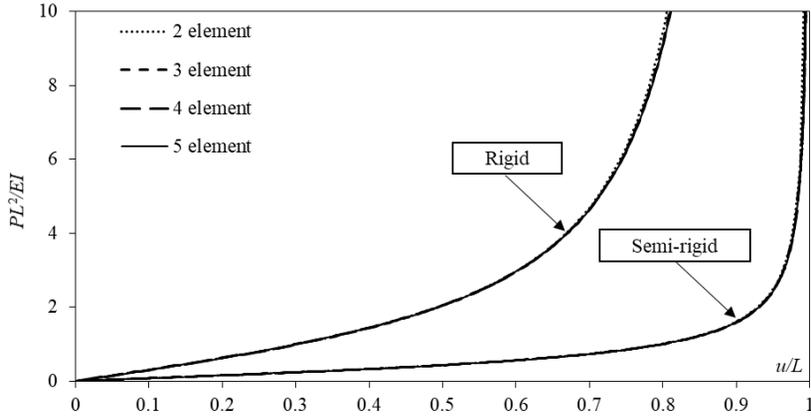


Figure 8. Equilibrium path equivalent to used proposed element quantity

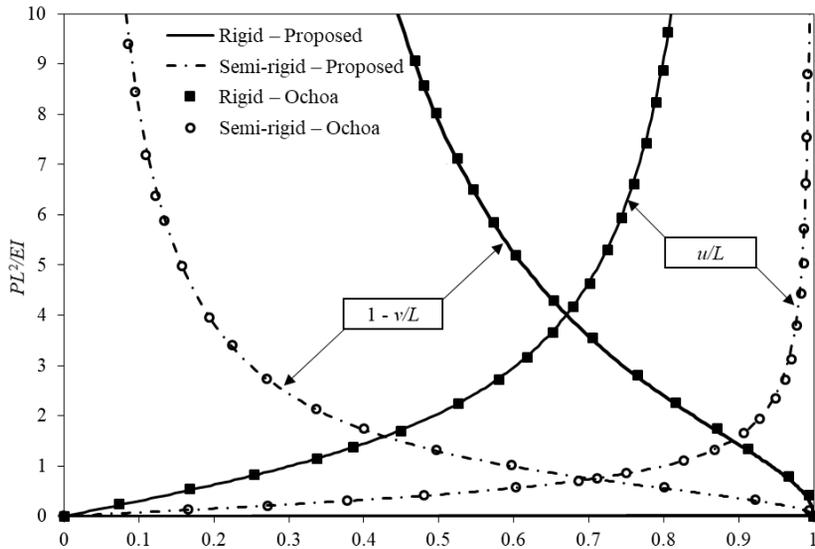


Figure 9. Load-displacement relationship at free end

### 3.3. William's toggle frame

Fig. 10 shows the properties of well-known William's toggle frame [12] where an analytical solution is given. This structure was then studied in three different boundary conditions including fixed,

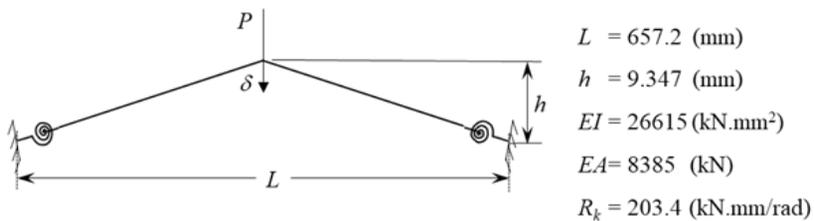


Figure 10. William's toggle frame

linear flexible and hinge by Tin-Loi and Misa [13]. Depicted in the Fig. 11 is the comparison of numerical results from using 1, 2 and 3 proposed elements, respectively. Again, two proposed elements are sufficient to achieve an acceptably converged result. As presented in Fig. 12, irrespective of boundary conditions, the obtained results reveal good convergence with those of Tin-Loi and Misa and William. Besides that, the program manages to tackle the “snap-through” behavior.

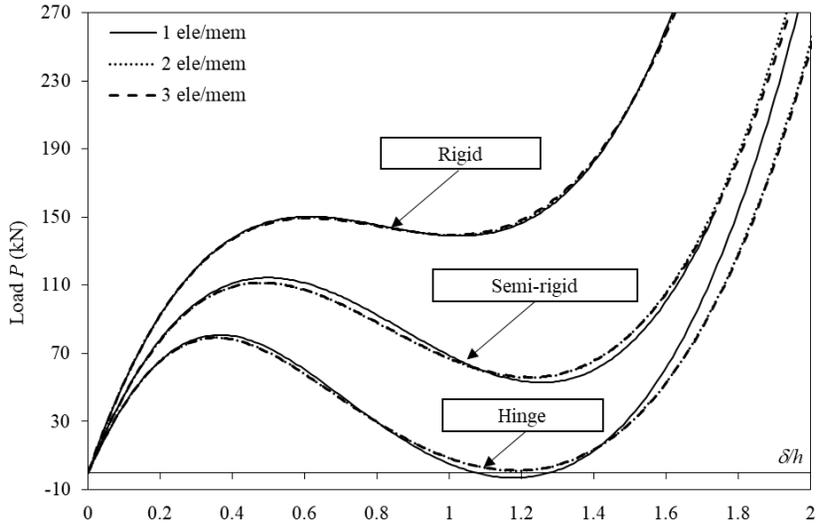


Figure 11. Number of proposed element versus convergence rate

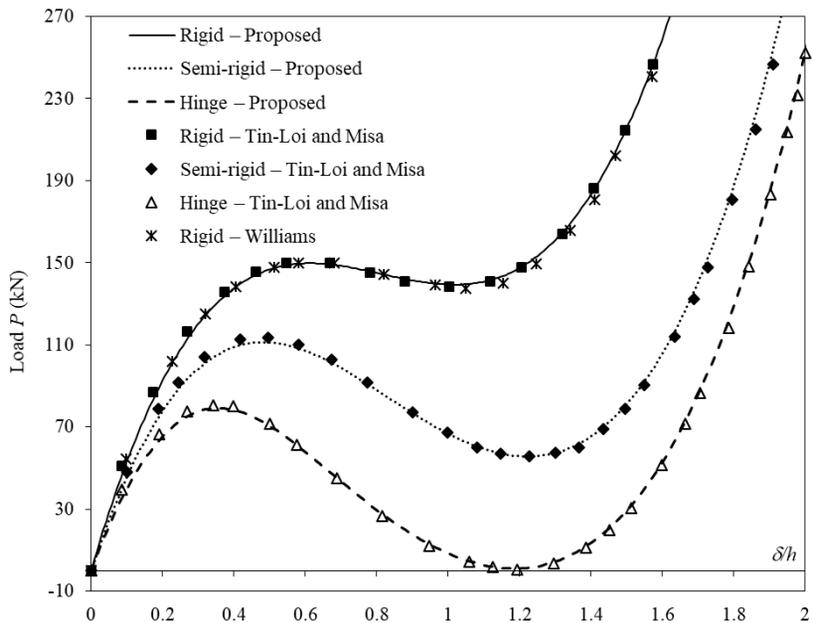


Figure 12. Load-deflection curve

#### 4. Conclusions

This study derives a co-rotational beam-column element for large-displacement elastic analysis of planar steel frames with flexible connections under static loads. Zero-length rotational springs with either linear or nonlinear moment-rotation relations are adopted to simulate the flexibility of beam-to-column connections. The modified spherical arc-length method coupled with the sign of displacement internal product is integrated into the MATLAB computer program to trace the load-displacement path regardless of the presence of “snap-back” or “snap-through” behavior. The results of numerical examples demonstrate the accuracy and effectiveness of the proposed element with the use of only two proposed elements in all examples.

#### Acknowledgments

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